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Memoir on the Substitution-Groups whose Degree does not Exceed Eight.

By G. A. Miller.

Part I.

On the Construction of all the Substitution-groups of a Small Degree.

From the time that Galois demonstrated that the theory of equations is based upon the theory of substitution-groups it has been observed that the determination of all the possible substitution-groups of a given degree is a problem of fundamental importance in algebra.* Although this problem has been studied by a large number of mathematicians, yet little progress has been made towards its complete solution. It is well known that all the groups of a given degree may be found by tentative methods, but for large degrees the number of trials that may be necessary becomes extremely large.

During recent years a large number of substitution-groups, which were usually found by tentative processes, have been published. All the possible groups whose degree is less than eleven have been determined, and the transitive groups, especially the primitive ones, have been determined for considerably larger degrees.† In what follows we shall aim to give enough of the general theory of group construction to find all the possible groups whose degree does not exceed eight without any tentative processes.

In the earliest work that devotes considerable attention to substitution-groups, *Teoria generale delle equazioni, in cui si dimostra impossibile la soluzioni algebraica delle equazioni generali di grado superiore al quarto*, di Paolo Ruffini,‡

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*Cf. Serret, "Algèbre supérieure" (1866), p. 256.

†Jordan, Comptes rendus, vol. LXXV, 1872, p. 175-7. Miller, American Journal, vol. XX, 1898, p. 229.

‡A review of the group theory that is contained in this work is given by Burkhardt, Schlömlich Zeitschrift, 1892, supplement, p. 159.

Bologna, 1799, we find that these groups are divided into two classes, viz. the transitive and the intransitive groups. The former of these classes is subdivided into the primitive and the imprimitive groups. These divisions have been adopted by all subsequent writers, and they will be employed in what follows.

When one of two substitution-groups can be transformed into the other, each of the two groups is called the transform of the other. In the enumeration of the possible substitution-groups, all of these transform groups are considered to be identical. Two substitution-groups are distinct or different when it is impossible to transform the one into the other.* Two or more different substitution-groups may represent the same abstract group.

It may happen that the substitutions of a group are completely determined by those of a subgroup. E. g. the substitutions of the non-cyclical regular group of order 6 are completely determined by those of its subgroup of order 3, while those of the regular cyclical group of this order are not fully determined by this subgroup, since there are six substitutions of degree and order 6 whose squares are contained in the given subgroup of order 3.

If the substitutions of a transitive group, which is known as an abstract group and whose order is composite, are completely determined by those of each one of its subgroups besides identity, the group must be regular and contain no substitution whose order is less than the order of the group divided by 2. Hence there are only two such groups, viz. the two regular groups of order 4. It is evident that the substitutions of an intransitive group are never completely determined by those of each one of its subgroups.

As an illustration of the importance of knowing whether all the substitutions of a group are completely determined by those of a given one of its subgroups, we may prove that there are no more than one group of degree eight and order 1344. Such a group must contain the doubly transitive simple group of degree seven and order $1344 \div 8 = 168$, and hence it must be triply transitive. It must, therefore, contain an intransitive subgroup of order 48 whose transitive constituents are of degrees two, six and whose transitive subgroup of order 24 and degree six is positive.† The substitutions of this subgroup of order 24 determine all the substitutions of the transitive group of order 48 and degree six, since they do not form a self-conjugate subgroup of a larger group of this degree;[*] each of these groups of order 1344 must, therefore, contain substitutions of degree eight which are fully determined by the substitutions of the given subgroup of degree seven and hence there cannot be more than one such group.

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* Cf. Burnside, Messenger of Mathematics, vol. XXVIII (1898), p. 103.

† Miller, Bulletin of the American Mathematical Society, vol. IV, 1898, p. 140.

§1.—*Intransitive Groups.*

An intransitive group contains two or more transitive constituents. Since the degree of each of these constituents is less than the degree of the group, we may suppose that the transitive constituent groups of the required intransitive groups are known. The first problem which we shall consider may be stated as follows: To determine all the possible substitution-groups that may be formed by combining intransitively a given system of transitive groups, two groups being considered as identical whenever one can be transformed into the other by means of some substitution.[†]

If an intransitive group contains more than two transitive constituents, any number of these constituents form a group which has an $1, \alpha$ isomorphism to the entire group and an α_1, α_2 isomorphism to the group formed by the remaining constituent or constituents. All such intransitive groups may therefore be constructed by establishing isomorphisms between the intransitive groups that contain all except one of the constituents of the required groups and the transitive group which is the remaining constituent For instance, to find all the possible intransitive groups of degree ten whose constituents are of degrees four, three, three, we have only to consider the isomorphisms between the intransitive groups of degree seven that involve transitive constituents of degrees four, three and the transitive groups of degree three. We can also obtain all these groups by considering all the possible isomorphisms between the intransitive groups of degree six that involve two transitive constituents of degree three and the transitive groups of degree four.

The form in which a group can be represented as an intransitive substitution-group depends upon its abstract group properties. It must contain at least one transitive constituent whose order is equal to the order of the entire group unless it contains two or more self-conjugate subgroups (differing from identity) which have only identity in common. If it contains two such self-conjugate subgroups, it can always be represented as an intransitive group which involves no transitive constituent whose order is equal to the order of the group. As such transitive constituents we may use the quotient groups (represented as substitution-groups) with respect to any two of the given self-conjugate subgroups. Since these remarks apply to the constituent groups as well as to the entire intransitive group, it follows that the form in which a group is represented as an intransitive substitution-group frequently throws light on the properties of the group.

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[*] When a self-conjugate subgroup of a group of degree n is not contained in any larger group of this degree as a self-conjugate subgroup, its substitutions must determine all the substitutions of the group, but the converse is not always true.

[†] Cf. Bolza, American Journal of Mathematics, vol. XI, 1889, pp. 195–214.

Let $G^{m_1}_{g_1}$ and $G^{m_2}_{g_2}$ represent two substitution-groups of degrees m_1 and m_2, and of orders g_1 and g_2 respectively, and suppose that none of the m_1 letters in the first of these two groups is found in the second. If we multiply each of the substitutions of one of these groups by all of the substitutions of the other, we obtain an intransitive group of degree $m_1 + m_2 = n$ and of order $g_1 g_2 = g$. This group G^n_g is said to be the *direct product*[*] of $G^{m_1}_{g_1}$ and $G^{m_2}_{g_2}$. It is evident that any abstract group which is the direct product of two groups may be represented as an intransitive substitution-group in which these subgroups are transitive and do not contain a common element.

The groups $G^{m_1}_{g_1}$ and $G^{m_2}_{g_2}$ are self-conjugate subgroups of G^n_g and they contain no common substitution besides identity. If both of these subgroups are simple and if $g_1 \neq g_2$ or if at least one of the subgroups is non-Abelian, G^n_g does not contain any other self-conjugate subgroup besides identity. In general, if k_1 and k_2 represent the numbers of the self-conjugate subgroups of $G^{m_1}_{g_1}$ and $G^{m_2}_{g_2}$ respectively, then will G^n_g contain just $(k_1 + 1)(k_2 + 1) - 1$ self-conjugate subgroups that are the direct products of a group contained in each of the given constituents. In this enumeration identity is considered a self-conjugate subgroup, but the entire group is not included in this term. Other self-conjugate subgroups of G^n_g can exist only when a quotient group of one of the given transitive constituents, or one of its subgroups, is simply isomorphic to a quotient group of the other or one of its subgroups, and when all the operators of each constituent are commutative to each of the operators of this quotient group. It is evident that these conditions are sufficient as well as necessary. Somewhat similar remarks apply to all the possible subgroups of G^n_g.[†] We shall incidentally consider some of these subgroups in what follows.

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[*] Cf. Hölder, Mathematische Annalen, vol. XLIII, p. 330.

[†] Cf. Klein-Fricke, Modulfunktionen, vol. I, 1890, p. 403.

Since each of the substitutions of one of the two subgroups $G_{g_1}^{m_1}$, $G_{g_2}^{m_2}$ is commutative to every substitution of the other, the order of any substitution of G_g^n is the lowest common multiple of the orders of its constituents that belong to these subgroups. In particular, it follows that the necessary and sufficient condition that G_g^n is cyclical is that g_1, g_2 are prime to each other, and that the given subgroups of these orders are cyclical. G_g^n is evidently Abelian when the two given subgroups have this property. These remarks clearly apply also to the case when $G_{g_1}^{m_1}$ and $G_{g_2}^{m_2}$ represent the same group written in different sets of letters. In this case G_g^n is said to be the square of one of these subgroups.

The intransitive group of order $g_1 g_2$ which we have just considered can always be constructed regardless of the particular groups which $G_{g_1}^{m_1}$ and $G_{g_2}^{m_2}$ may represent. Other intransitive groups having these constituents can be constructed only when certain conditions are satisfied, and when they are possible they must be subgroups of G_g^n. We proceed to consider the necessary conditions.

Suppose that G_m^n is any intransitive group with the constituents $G_{g_1}^{m_1}$ and $G_{g_2}^{m_2}$. All the substitutions of $G_{g_1}^{m_1}$ which correspond to a self-conjugate subgroup of $G_{g_2}^{m_2}$ must form a self-conjugate subgroup of $G_{g_1}^{m_1}$. In particular, the substitutions of one of the transitive constituents which correspond to identity in the other transitive constituent must form a self-conjugate subgroup. It is also evident that the number of the substitutions in one of these constituents which correspond to identity in the other constituent is equal to the number that correspond to any other substitution in the second constituent. Hence we see that if H_1 is the self-conjugate subgroup of $G_{g_1}^{m_1}$ that corresponds to identity in $G_{g_2}^{m_2}$ and if H_2 is the self-conjugate subgroup of $G_{g_2}^{m_2}$ that corresponds to identity in $G_{g_1}^{m_1}$, then must the quotient group of $G_{g_1}^{m_1}$ with respect to H_1 be simply isomorphic to the quotient group of $G_{g_2}^{m_2}$ with respect to H_2 and m must equal $h_1 g_2 = h_2 g_1$, h_1 and h_2 being the orders of H_1 and H_2 respectively.

The construction of the intransitive groups which contain two given constituents is thus reduced to two cases, viz.: 1), forming the direct product of the substitutions of the constituents, and 2), forming all the possible simple isomorphisms between the constituent groups and their quotient groups, which lead to substitution-groups that cannot be transformed into each other. The former of these cases is very simple and requires no further attention, while the latter is generally much more complex whenever such groups are possible.

We have observed that a necessary condition to construct an intransitive group besides G_g^n with the given constituents is that their quotient groups with respect to H_1 and H_2 (where H_1 or H_2 or both of them may be identity) are the same abstract group. This condition is evidently sufficient as well as necessary for the construction of G_m^n, where m is a factor of g and less than g. When the order of the given quotient group exceeds 2, it has more than one simple isomorphism to itself. In this case it is necessary to consider the intransitive groups which correspond to all the possible simple isomorphisms of the given quotient group to itself. Since all the cogredient isomorphisms lead to the same group, the total number of these intransitive substitution-groups cannot exceed the quotient of the total number of the simple isomorphisms of the quotient group to itself divided by the number of the cogredient ones.

If the given quotient group is cyclical, there is clearly only one group with respect to the given self-conjugate subgroups. If this quotient group is complete,* it admits only cogredient simple isomorphisms to itself, and hence there cannot be more than one group for H_1 and H_2. When either H_1 or H_2 is identity and the corresponding constituent group is regular, there cannot be more than one group, since every simple isomorphism of a regular group to itself can be obtained by transforming the group by certain operators.†

In general, we may obtain all the possible intransitive groups with respect to the self-conjugate subgroups H_1 and H_2 in the following manner: First determine the group of isomorphisms of the given quotient group and find the operators of this group of isomorphisms that correspond to all the possible conjugates of a given intransitive group. If these do not include all the operators of the group of isomorphisms, form an isomorphism that corresponds to another operator of this group. We thus obtain a second intransitive group, and all the conjugates of this group correspond to operators of the group of isomorphisms which are distinct from those that correspond to the first intransitive group. If this does not exhaust all the operators of the group of isomorphisms of the given quotient group, we may form a third intransitive group, etc. We continue this process until all the operators of the group of isomorphisms are exhausted.

When the two transitive constituents are conjugate, the set of operators of the group of isomorphisms which corresponds to the conjugates of any intransitive group must include all the inverse operators of the set, since the inverse of such an operator simply indicates an interchange of the constituents. Hence the number of the intransitive groups that can be formed by making two conjugate groups isomorphic cannot equal the order of the group of isomorphisms of the corresponding quotient group unless this group of isomorphisms includes no operator whose order exceeds two. For instance, there cannot be more than five such groups when the given group of isomorphisms is the symmetric group of degree three. This number is reached when all the possible 2, 2 isomorphisms are established between $(abcd)_8$ and $(efgh)_8$.*

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* Hölder, Mathematische Annalen, vol. XLVI, 1895, p. 325.

† Cf. Netto, "Theory of Substitutions," 1892, p. 110.

Intransitive groups containing only transitive constituents of degree two.

The method which has been explained may be employed to construct all the possible intransitive groups of any given degree. In some cases it is desirable to employ other methods. To illustrate one of these, we shall employ it to find all the possible intransitive groups of degree eight that contain four systems of intransitivity. The average number of elements in all the substitutions of such a group is $8 - 4 = 4$† and the positive substitutions must be of the form $ab \, . \, cd$, or $ab \, . \, cd \, . \, ef \, . \, gh$, while the negative substitutions must have one of the two following forms: ab, $ab \, . \, cd \, . \, ef$.

There is evidently only one such group of order 2, which may be represented by $(ab \, . \, cd \, . \, ef \, . \, gh)$. The average number of letters in the three substitutions, differing from identity, of a group of order 4 must be $16 \div 3 = 5\frac{1}{3}$. If all of these are positive, two of them must be of degree four and one of degree eight. Since the two of degree four must generate the entire group, they cannot contain any common element. Hence there is only one positive group of order 4 that contains the given systems of intransitivity. If a group of order 4 contains two negative substitutions it must also contain a positive substitution that is not identity. If the degree of this positive substitution is 4, the two negative substitutions must involve 12 letters. Hence each of these negative substitutions must be of degree six and have two elements in common with the given positive substitution. There is, therefore, only one such group of order 4.

If the degree of the positive substitution is eight, the two negative substitutions must include eight elements. As one of them must be a single transposition, and the given positive substitution is symmetric in regard to all the transpositions, there is only one group of this kind. We have now considered all the possible cases, and have found that there are three groups of degree eight and order 4 that contain four systems of intransitivity. Two of these include negative substitutions.

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* Cf. Bulletin of the New York Mathematical Society, vol. III (1894), p. 168.

† Frobenius, Crelle, vol. CI, p. 287; cf. Miller, Bulletin of the American Mathematical Society, vol. II, 1895, p. 75.

There is clearly only one group of order 16 that contains the given systems of intransitivity, and all the groups of degree eight that contain four systems of intransitivity must be contained in this group of order 16. Since this group contains negative substitutions, there can be only one positive group of order 8 that involves the given systems. If a group of order 8 contains four negative substitutions, its positive subgroup of order 4 must be either of degree six or of degree eight. In the former case the negative substitutions must include $32-12=20$ elements, and in the latter case they must include $32-16=16$ elements. Hence at least one of them must be a single transposition.

When the positive subgroup of order 4 is of degree six, this transposition must involve the other two elements, and there is only one possible group. In case this subgroup is the given positive group of degree eight and order 4, the transposition must be included in one of the substitutions of degree four. As these two substitutions can be transformed into each other, and such a transforming substitution must transform the given subgroup of order 4 into itself, there is only one group in this case. Hence there are 3 groups of order 8.

We have now considered all the possible groups of degree eight that involve 4 systems of intransitivity and found that there is one group of each of the orders 2 and 16, and that there are three groups of each of the orders 4 and 8. The total number of the possible substitution groups of degrees six and four that contain three and two systems of intransitivity respectively, may be found by similar but briefer considerations. This method can evidently also be employed in regard to the groups of larger degrees.

It may be observed that for small degrees the numbers of the groups involving only systems of intransitivity of degree two follow the law of the coefficients of the binomial expansion; i. e.,

DEGREE.	ORDERS.				
	2	4	8	16	32
4	1	1			
6	1	2	1		
8	1	3	3	1	
10	1	4	6	4	1

This rule does not apply to the groups of a higher degree, since there are six such groups of degree twelve and order 4.*

§2.—*Imprimitive Groups.*

The substitutions of an imprimitive group must permute all its possible systems of imprimitivity according to a transitive substitution group. Hence each system must contain the same number of elements; i. e., it is impossible to construct an imprimitive group of a prime degree. If the transitive group, according to which the systems of imprimitivity are permuted, is itself imprimitive, it is possible to combine these systems until we obtain systems which are permuted according to a primitive group.† Hence every imprimitive group must have an α, 1 isomorphism to some primitive group which corresponds to the permutations of one set of its systems of imprimitivity when its substitutions are transformed by every substitution of the group.

The subgroup of order α which corresponds to identity in the given primitive group must be self-conjugate. If its order exceeds unity, it must be intransitive, and its systems of intransitivity are also systems of imprimitivity of the given group. It should, however, be observed that the latter systems of imprimitivity are not necessarily permuted, by all the substitutions of the group, according to the given isomorphic primitive group. In case the systems are permuted according to an imprimitive group, they may always be united into larger

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***Philosophical Magazine, vol. XLI, 1896, p. 435.**

†Jordan, "Traité des Substitutions," 1870, p. 399.

systems which are permuted according to the given isomorphic primitive group.* If, however, this primitive group is regular, the systems of intransitivity of the given subgroup of order α must coincide with the former systems of imprimitivity, since all the systems of intransitivity of the given subgroup must be permuted transitively by all the substitutions of the given imprimitive group.

Every group of a finite order may be represented as a regular substitution group, and every regular group of composite order is imprimitive.† Hence every group of a finite order, with the exception of the trivial case where the order is a prime number, may be represented as an imprimitive substitution-group whose degree is equal to its order. This representation is unique; i. e., there is one and only one regular substitution-group which is simply isomorphic to any given group of a finite order. Some groups can also be represented as non-regular, imprimitive groups. The properties of these groups will be considered later.

It is convenient to have a special name for the subgroup of order α which contains all the substitutions of the group that do not permute any of the systems of imprimitivity. It has been called the *base* or the *head* of the group. We shall generally use the latter term. All the substitutions of an imprimitive group that are not contained in the head are said to form the *tail* of the group. We have already observed that each of the transitive constituents of the head must be of the same degree. Since all of them are transformed transitively by the substitutions of the imprimitive group, they must be similar to each other.

It is well known that every non-Abelian regular group is conjugate to a group containing the same letters whose substitutions are commutative to every substitution of the regular group. Each substitution of one of these groups determines systems of imprimitivity of the other group. The systems of imprimitivity obtained in this way do not necessarily include all the possible systems. When a regular group of composite order is Abelian, its own substitutions determine different systems of imprimitivity. Hence it follows that the elements of a group can sometimes be arranged into systems of imprimitivity in a large number of ways.

1.—*The Head of an Imprimitive Group.*

It is sometimes impossible to find any systems of imprimitivity of a given imprimitive group, which are not permuted among themselves by every substitution of the group, that differs from unity. Such imprimitive groups occur for degrees twelve and fourteen and for higher degrees, but not for any lower degrees. In these cases we say that the only possible head of the group is identity. In what follows we shall consider the heads whose orders exceed unity. We have already observed that the transitive constituents of these heads are similar groups, i. e. the heads are obtained by establishing some isomorphism between a transitive group written in two or more different sets of elements. It remains only to inquire into the nature of this isomorphism.

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* Cf. Miller, Quarterly Journal of Mathematics, vol. XXIX, p. 235.

† Jordan, "Traité des Substitutions," 1870, p. 60.

The simplest possible cases are those in which we merely form the product of the groups obtained by writing a given transitive group in different sets of letters or establish a simple isomorphism between the corresponding substitutions of these groups. These two heads are always possible, regardless of the number of systems of imprimitivity. Other heads containing a given transitive constituent can be constructed only when this transitive constituent satisfies certain conditions.

If a transitive constituent of a head is compound, it is easy to form other heads by merely multiplying the divisions with respect to the same self-conjugate subgroup in the different sets of letters. The heads thus obtained are all symmetrical with respect to the transitive constituents. When the primitive group, according to which the systems of imprimitivity are permuted, is not symmetrical, it is simply necessary that the head allows the permutations indicated by this primitive group. It is thus frequently possible to construct imprimitive groups whose heads do not permit the symmetric permutation of its systems of intransitivity.*

2.—*Groups containing Two Systems of Imprimitivity.*

One-half of the substitutions of such a group must transform each of these systems into itself. These must form an intransitive group which differs from unity, since the degree of an imprimitive group cannot be less than four. Let G_g^n be a transitive constituent of the head. We shall first suppose that $g = n!$ and that the head is the product of its constituents. By adding to this head a substitution of order 2 that merely interchanges the corresponding elements of its constituents we obtain an imprimitive group of order $2(n!)^2$ This group evidently includes all the imprimitive groups of degree $2n$ that contain two systems of imprimitivity, and it includes no other transitive group of this degree. Hence each of the substitutions that is contained in the tail of one of these imprimitive groups is of degree $2n$, and it involves only negative cycles. It must clearly satisfy the following three conditions: (α), interchange the systems of intransitivity of the head; (β), transform the head into itself; (γ), its square must be contained in the head. These conditions are sufficient as well as necessary. The factors of composition of these imprimitive groups are clearly obtained by adding 2 to the factors of composition of the head.

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* **Cf. Miller, Quarterly Journal of Mathematics, 1896, vol. XXVIII, p. 197.**

We proceed to establish theorems which are useful in constructing the imprimitive groups that involve only two systems of imprimitivity. Several of these theorems are included in the more general theorems of the following sections, and are given here for the purpose of leading up to them. We shall represent the n elements of the two systems by $a_1, a_2, a_3, \ldots, a_n$ and $a'_1, a'_2, a'_3, \ldots, a'_n$ respectively, and we shall use s_α, s'_α to represent the same substitution in the two systems, the elements of s_α belonging to the first system and those of s'_α to the second. Similarly we shall employ G, G' respectively to represent the same group in the two systems. The substitution of order 2 which merely permutes the corresponding elements of the two systems we shall represent by t. Hence we have $t = a_1a'_1 . a_2a'_2 . a_3a'_3 \ldots . a_na'_n$, $t^{-1}s_\alpha t = s'_\alpha$, $t^{-1}Gt = G'$.

Theorem I. *If one of two transitive constituents of the head of an imprimitive group is G, and if H is the largest group involving the same elements as G that transforms G into itself, this imprimitive group is included in the imprimitive group generated by H, H', t.*

We may choose the two transitive constituents of the head in such a way that t transforms one into the other. Since the remaining generator of the imprimitive group in question must also have this property, the factor which must be multiplied into t to obtain this generator, must transform each of the given constituents of the head into itself. Hence this factor must be contained in the product of H and H'.

Theorem II. *The squares of the substitutions of the tail of the smaller imprimitive group of the preceding theorems are transforms, with the respect to substitutions of H, of the substitutions obtained by making H simply isomorphic to H' in such a way thas t is commutative to every substitution of this group. When H coincides with G, all these transforms are squares of substitutions in the tail of the group generated by H, H', t.*

We may represent any substitution of the tail of the given group by $s_\alpha s'_\beta t$, where s_α, s'_β belong to H, H' respectively according to the preceding theorem. Hence

$$(s_\alpha s'_\beta t)^2 = s_\alpha s_\beta s'_\beta s'_\alpha = s_\beta^{-1} s_\beta s_\alpha s'_\beta s'_\alpha s_\beta\,.$$

Since $s_\beta s_\alpha s'_\beta s'_\alpha$ is in the given simple isomorphisms of H to H' and s_β belongs to H, the first part of the theorem is proved. It remains to show that the squares of the substitutions in the tail of H, H', t give all the transforms with respect to substitutions of H of the substitutions of the group obtained by making H simply isomorphic to H' in such a way that identical substitutions correspond whenever H and G are identical. In other words, that it is possible to choose s_α, s_β in such a manner as to satisfy the equation $s_\alpha s_\beta s'_\beta s'_\alpha = s_\gamma^{-1} s_\delta s'_\delta s_\gamma$, where s_δ and s_γ are any given substitutions of H. If we let $s_\alpha = s_\gamma^{-1} s_\delta$ and $s_\beta = s_\gamma$, this equation is satisfied and the proof is complete.

Theorem III. *If G is any transitive group, there is always one and only one imprimitive group whose head is the direct product of G and G'.*

Let $s_\alpha s'_\beta t$ be any substitution that may be added to this head to generate an imprimitive group. Since the square of this substitution must be found in the head, we have $(s_\alpha s'_\beta t)^2 = s_\alpha s_\beta s'_\beta s'_\alpha =$ some substitution of the head. Hence either, both or neither of the substitutions s_α, s_β belong to G. In the former case we evidently have an imprimitive group; in the latter case it is necessary that s_α, s_β correspond to the inverse operators of the quotient group of H with respect to G, H being the largest group (involving the same elements as G) that transforms G into itself. Hence $s_\beta = s_\alpha^{-1} s$ where s is some substitution of G.

If we transform t by the inverse of s_α we obtain $s_\alpha t s_\alpha^{-1} = s_\alpha t s_\alpha^{-1} t t = s_\alpha s'^{-1}_\alpha t$. Since s_α transforms the given head into itself, we observe that the groups obtained in the second case of the preceding paragraph are conjugate to that of the first case. From this it follows that all the groups imprimitive that contain the given head are conjugate.

Theorem IV. *If G has a self-conjugate subgroup which is also a self-conjugate subgroup of H, and if no two of the m divisions of H with respect to this self-conjugate subgroup are similar, then there are $g \div k = l$ imprimitive groups having for a common head the group obtained by the simplest k, k isomorphism between G and G', where g and k are the order of G and the given self-conjugate subgroup respectively.*

Let $s_1, s_2, \ldots\ldots, s_n$ satisfy the condition that one of them belongs to each one of the given divisions of H. Each one of the substitutions $s_1 t$, $s_2 t$, $s_3 t, \ldots\ldots, s_m t$

transforms the given head into itself, but only l of them (say the first l) have their squares in this head. The l substitutions combined separately to the head lead to l imprimitive groups. These groups must be distinct because the squares of the substitutions of the tails are not similar. The remaining substitutions of the product of H and H' may be obtained by multiplying the given head into each of the following substitutions:

$$s_\beta s'_\beta s_\alpha t, \quad \alpha = 1, 2, \ldots\ldots, m, \quad \beta = l + 1, 2l + 1, \ldots\ldots, m - l + 1.$$

From
$$s_\beta^{-1} s_\beta s'_\beta t s_\beta = s'^2_\beta s_\alpha t$$

it follows that these are conjugate with respect to substitutions that transform the given head into itself, to the substitutions considered above. Hence l is the total number of the imprimitive groups containing the given head.

Theorem V. *There is only one imprimitive group whose head is obtained by making the symmetric group whose degree is not two or six simply isomorphic to itself. If the degree of the symmetric group is two or six, there are two such groups.*

Since the symmetric group whose degree is not two or six, is a complete group,* we can form only one intransitive group by making such a symmetric group simply isomorphic to itself. We may suppose that in this intransitive group the corresponding substitutions are identical in the two systems. If we add to this group a substitution of order 2 which simply interchanges the corresponding elements of the two systems (t), we obtain an imprimitive group whose order is twice the order of the given symmetric group.

Suppose that a different tail to the given head was generated by the head and the substitution s. This substitution must interchange the two systems of intransitivity of the head. Hence we have

$$st = s_1 \text{ or } s = s_1 t,$$

where s_1 must transform the given head into itself without permuting its systems of intransitivity, hence the second tail would contain a substitution of the form $s_2 t$, where s_2 involves only elements from one of the systems of intransitivity of the head. Since the head is a complete group, this is impossible, unless $s_2 \equiv 1$. In this case we have the group given above. As an abstract group this is the direct product of the given symmetric group, and an operator of order 2.

When the degree of the symmetric group is six, we can construct two intransitive groups by making it simply isomorphic to itself. To each one of the heads obtained in this way, we can add only one tail for the same reason as was given above. Hence there are just two such groups.* When the degree is two, we evidently obtain the two regular groups of order 4.

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*Hölder, Mathematische Annalen, 1895, vol. XLVI, pp. 345 and 325.

Theorem VI. *There are only two imprimitive groups whose common head is the group obtained by making the alternating group whose degree is not six simply isomorphic to itself. If the degree of the alternating group is six, there are four such groups.*

When the degree of the alternating group differs from six, we can obtain only one intransitive group by making it simply isomorphic to itself.† We may, therefore, suppose that t is commutative to every substitution of the given head. It is evident that t and the given head generate an imprimitive group which is simply isomorphic to the direct product of the given alternating group and an operator of order 2.

As under the preceding theorem, we may suppose that s and this head generate another imprimitive group, and we find by the same process of reasoning that s_1 must transform the given head into itself without interchanging its systems of intransitivity. We can, however, not assume that s_2 involves only elements from one of the systems; but, since the symmetric group of any degree larger than three contains no substitution besides identity that is commutative to each one of the substitutions in its alternating subgroup, we must assume that t is commutative to s_2. If s_2 is not found in the given head, all its possible values may be found by multiplying this head by s_2. Hence there are no more than two such groups.

That the two possible imprimitive groups just found are really distinct, follows directly from the fact that the latter is simply isomorphic to the symmetric group while the former does not have this property. Some of these arguments do not apply to the case when the degree of the alternating group is three, but in this case we obtain the well-known two regular groups of degree six. When the degree of the given alternating group is six, it is possible to construct two intransitive groups by making it simply isomorphic to itself. To each one of these heads we may add two tails and thus obtain the given four imprimitive groups. These groups are explained in connection with the groups whose head is the symmetric group to which references are given above.

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* Miller, Bulletin of the American Math. Society, 1895, vol. I, p. 258.

† Hölder, Mathematische Annalen, 1895, vol. XLVI, p. 340; cf. Miller, loc. cit.

From the preceding theorems it follows that if one of the two transitive constituents of the head is the symmetric group of degree n, where n equals 3, 5 or exceeds 6, there are only 4 possible imprimitive groups. According to theorems III and V, there is only one such group for each of the two heads obtained by multiplying the two transitive constituents of the head and by making them simply isomorphic. According to theorem IV, there are two such groups whose head consists of the positive substitutions in the product of its constituents. When n is greater than 6 or $n = 3, 5$, we can evidently form no other head with the given constituents, and hence no additional imprimitive group of this type.

When $n = 2$, there are evidently only two possible heads. When the head is of order two, we may obtain the two regular groups of order 4, and when it is of order 4, there is only one group according to theorem III. Hence there are only three imprimitive groups of degree four. When $n = 4$, the given constituent group is isomorphic to the symmetric group of degree three. We thus obtain one more head than occurs in the general case. It follows directly from theorem V that we can add only one tail to this head. Hence there are five imprimitive groups of degree eight in which a transitive constituent of the head is the symmetric group of degree four. When $n = 6$, there is one head that does not occur in the general case. This case was considered under theorem V. Hence there are five imprimitive groups of degree twelve in which one of the transitive constituents of the head is the symmetric group of degree six. This completes the study of the imprimitive groups of two systems whose heads contain the symmetric group as a transitive constituent.

We shall now consider all the possible imprimitive groups of degree $2n$ whose heads contain the alternating group of degree n as a transitive constituent. For $n = 2$ there evidently is no such group. When $n = 3$, there are the two regular groups of order 6 and the group of order 18 whose head is the product of two alternating groups of degree three. When $n = 4$, we may construct three heads, since the alternating group of degree four is isomorphic to the alternating group of degree three. The groups which contain either one of two of these heads have been considered under the preceding theorems. It remains to consider what tails may be added to the head of order 48.

If we represent the substitution of order two which merely interchanges the corresponding elements of the two systems by t, we observe that t and this head generate an imprimitive group of order 96. All the other possible tails can be generated by st, where s transforms the head into itself without permuting its systems of intransitivity and $(st)^2$ is contained in the head. Since we might use the product of any substitution in the head into st, in place of st we may suppose that s either belongs to the first system of the head or that it is a particular substitution of the second system. Hence we can add only two tails that are not conjugate to this head, i. e. there are five imprimitive groups of degree eight whose heads contain the alternating group of degree four as a transitive constituent.

When $n = 6$, we have seen that there are also five imprimitive groups of the type in question. For all the values of n which differ from those that have just been considered, we can construct only the two heads which are included in theorems III and VI. Hence *there are, in general, only three imprimitive groups of degree* $2n$ *whose heads contain the alternating group of degree* n *as a transitive constituent;* when $n = 4$ or 6, *there are five such groups.*

All of the preceding groups are evidently also distinct as abstract groups, and when n exceeds 4, one of the factors of composition must be $n! \div 2$, hence all of these groups are insolvable whenever their degree exceeds eight. When their degree does not exceed eight, they are evidently solvable.

3.—*Groups containing more than Two Systems of Imprimitivity.*

We have seen that every imprimitive group must contain systems of imprimitivity which its substitutions transform according to an isomorphic primitive group (P). The imprimitive group is either simply or multiply isomorphic to P. In the former case it contains no substitution besides identity that transforms each of these systems of imprimitivity into itself, while in the latter case it must contain an intransitive self-conjugate subgroup which is composed of all the substitutions of the group that transform each of the given systems of imprimitivity into itself and which corresponds to identity of P in the given isomorphisms. If the systems of intransitivity of this self-conjugate subgroup differ from the given systems of imprimitivity, they may be united as units in such a manner as to form these systems.

If the required imprimitive group (G) is simply isomorphic to P, the latter (which is supposed to be known) must contain non-maximal subgroups that include no self-conjugate subgroup of P besides identity, and whose order is the quotient obtained by dividing the order of P by the degree of G. The required imprimitive groups can therefore not be constructed unless P contains at least one system of conjugate subgroups that satisfy the given condition, and in this case the number of the possible imprimitive groups is known.*

Every self-conjugate subgroup of P must be transitive, since a primitive group cannot contain an intransitive self-conjugate subgroup. If n and α represent respectively the degree of P and the number of elements in each of the systems of imprimitivity of G, and if P contains regular subgroups, each of the corresponding subgroups of G will contain α systems of intransitivity. If one of these regular subgroups of G is self-conjugate, G must also contain α systems of imprimitivity, viz. the systems of intransitivity of the subgroup that corresponds to the given regular subgroup of P. For instance, each of the imprimitive groups of degrees six and eight that are simply isomorphic to the symmetric groups of degrees three and four respectively must also contain two systems of imprimitivity.

When P is contained in the metacyclic group of degree p, G must contain α systems of imprimitivity as well as p systems, and when P is the symmetric group of degree $n > 2$, there is always one and only one imprimitive group of degree $2n$ that contains n systems of imprimitivity, and is simply isomorphic to P. This group contains also two systems of imprimitivity. The total number of the imprimitive groups that are simply isomorphic to the symmetric group of degree n when n is not equal to 6, is clearly equal to the total number of the substitution-groups whose degree does not exceed $n - 1$, excluding identity, increased by the number of the substitution-groups of degree n that are not maximal subgroups of the symmetric group of this degree.† We proceed to establish some general theorems with respect to the imprimitive groups containing more than two systems of imprimitivity and which are not simply isomorphic to the primitive group according to which these systems are permuted.

Theorem I. *If each of the systems of imprimitivity includes a prime number*

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* Jordan, "Traité des substitutions" (1870), p. 57; cf. Miller, Bulletin of the American Mathematical Society, vol. III (1897), p. 215.

† Dyck, Mathematische Annalen, vol. XXII (1882), p. 91; cf. Proceedings of the American Philosophical Society, vol. XXXVI, 1897, p. 208.

of elements, the elements of the transitive constituents of the head must be the systems of imprimitivity of the group.

Since these transitive constituents must also be systems of imprimitivity, each of them must be composed of the same number of elements. If they would not be the given systems of imprimitivity, these could be obtained by combining the degrees of some of the transitive constituents taken as units. This is impossible, since the given systems of imprimitivity are supposed to be prime.

Theorem II. *An imprimitive group of degree n whose systems of imprimitivity are transformed by all the substitutions of the group according to a primitive group and whose head is not identity, must transform the transitive constituents of this head either according to the given primitive group or according to an imprimitive group that is simply isomorphic to this primitive group and whose systems of imprimitivity are transformed according to this primitive group.*

The systems of intransitivity of the head must be transformed according to a transitive group whose order cannot differ from the order of the primitive group according to which the systems of imprimitivity of the group are transformed, since these systems of imprimitivity could not be permuted without permuting the systems of intransitivity of the head. By means of these theorems the construction of all the possible imprimitive groups of degree n whose head is not identity, is reduced to the construction of the groups that permute the systems of intransitivity of given self-conjugate intransitive subgroups according to known groups. E. g. to construct all such imprimitive groups of degree twelve it is only necessary to construct the groups that contain a self-conjugate subgroup involving two systems of intransitivity, those that contain self-conjugate subgroups involving three systems of intransitivity, those that contain self-conjugate subgroups involving four systems of intransitivity which are transformed according to the symmetric or the alternating group of degree four, and those involving six systems of intransitivity which are transformed according to a primitive group of degree six or according to the non-Abelian regular group of this degree. In the last case the group would also contain two systems of imprimitivity, and hence this does not need to be considered for this particular degree.

Theorem III. *If one of the m transitive constituents of the head of an imprimitive group is G, and if H is the largest group involving the same elements as G that transforms G into itself, this imprimitive group is included in the imprimitive group whose head is the direct product of mH's written in different systems of elements and* *whose tail permutes these H's in the same way as the given imprimitive group permutes the given G's.*

If t is any substitution in the tail of the required imprimitive group, it must possess the following properties: 1) transform the given head into itself, 2) permute its systems of intransitivity according to a given substitution, and 3) its first power that leaves all these systems of intransitivity unchanged must be contained in the head. Since the group whose head is the direct product of the mH's includes all the substitutions that satisfy the first and the second of these three conditions, it must include the required group.

Corollary. *When the head of an imprimitive group is the direct product of transitive groups which are not self-conjugate in any larger group of the same degree, there is only one imprimitive group that permutes the transitive constituents of the head according to a given group.*

Theorem IV. *There is only one imprimitive group whose head is the direct product of a system of m conjugate transitive groups written in different sets of elements and whose tail permutes all these transitive groups cyclically.**

According to the preceding theorem, such a group must be contained in the imprimitive group whose head is the direct product of mH's and whose tail is generated by a substitution (t) of order m which permutes the corresponding elements of the H's in order. Hence its tail is generated by

$$s_1^1 s_2^2 \ldots . s_m^m t,$$

where the upper index indicates the system and the lower the particular substitution of the system. Since the m^{th} power of this substitution is contained in the head of the required group, we have

$$s_1^1 s_2^1 s_3^1 \ldots . s_m^1 = h,$$

where h is some substitution in the first transitive constituent of the head. If this condition is satisfied, the group generated by the given head and $s_1^1 s_2^2 s_3^3 \ldots . s_m^m t$ is conjugate to the group generated by this head and t; i. e. there is only one such imprimitive group.

§3.—*Primitive Groups.*

Every transitive group of degree n contains $n \div \alpha$ conjugate subgroups of degree $n - \alpha$, each including all the substitutions of the group that do not involve a given element. The necessary and sufficient condition that the group is primitive is that each of these subgroups is maximal. If this condition is satisfied, $\alpha = 1$, but the converse is not always true, i. e. α may be equal to unity in an imprimitive and also in an intransitive group. The following method of constructing all the primitive groups of a small degree (n) is based very largely upon the given maximal subgroup of degree $n - 1$, which is supposed to be known. We shall first consider the case when n is a prime number (p).

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* Cf. Quarterly Journal of Mathematics, vol. XXVIII, 1896, p. 207,

Every transitive group of degree p is primitive and contains at least one subgroup of order p. If it contains only one such subgroup, it must be included in the metacyclic group of this degree. All the transitive subgroups of this group are self-conjugate and correspond to the subgroups contained in the cyclical group of order $p - 1$,[*] if we include identity and the entire group under the term subgroup. As a cyclical group contains one and only one subgroup whose order is any given divisor of the order of the cyclical group, there are just as many transitive groups of degree p that contains only one subgroup of order p as there are different divisors of $p - 1$. E. g. there are just four primitive groups of degree seven that contain only one subgroup of order 7. The orders of these groups are 7, 14, 21 and 42.

All the subgroups of order p of any primitive group of degree p generate a self-conjugate subgroup, which is a simple group.[†] This simple group cannot be self-conjugate in more than one group of degree p and of a given order, since the substitutions which transform a subgroup of order p into itself are completely determined by this subgroup. Hence it is very easy to determine all the primitive groups of degree p that contain a given simple group as self-conjugate subgroup. We have only to find the largest subgroup of the metacyclic group that contains any one of the substitutions of order p which is found in the given simple group and transforms this simple group into itself. If the order of this subgroup is α times the order of the subgroup of the same metacyclic group that is contained in the given simple group, then this simple group is self-conjugate in just as many groups of degree p as there are different divisors of α, and the generators of these groups are contained in the given simple group and the given subgroup of the metacyclic group. Our problem is therefore reduced to the determination of the simple groups of degree p whose order exceeds p.

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[*] Since p has primitive roots.

[†] Miller, Bulletin of the American Mathematical Society, vol. IV (1898), p. 139.

Each subgroup of order p that is contained in such a group must be transformed into itself by more than p and by less than $p(p-1)$ substitutions of the group. Hence the order of the required groups must be of the form

$$\alpha p(1+kp), \qquad 1<\alpha<p-1$$

α must also be a divisor of p — 1. Sylow has recently considered the case when $k=1$, and he demonstrated that the groups of degrees five, seven, eleven and of orders 60, 168, 660 respectively are the only groups of a prime degree whose order is $\frac{1}{2}p(p^2-1)$.*

If a primitive group (G) of degree n is only simply transitive, its maximal subgroup (G) of degree $n-1$ is intransitive, and if the order of one of its transitive constituents is divisible by a given prime number (p), the order of each of the other transitive constituents is divisible by the same prime number.† Hence we observe that the order of G_1 must be a prime number whenever one of its transitive constituents is of a prime order. In particular, if G_1 contains a transitive constituent of degree 2, the order of G_1 is 2 and the degree of G is a prime number,‡ i. e. G is contained in the metacylic group.

Since G_1 is a maximal group, none of its self-conjugate subgroups besides identity can be transformed into itself by any substitution that is not found in G_1. Hence we see that when G_1 contains a self-conjugate subgroup of (H) of degree $n-\alpha$, H must be intransitive and it must be transformed by substitutions of G into $\alpha-1$ other subgroups of G_1. When G_1 is Abelian or Hamiltonian, all its substitutions except identity must therefore be of degree $n-1$, and the order of G cannot exceed $\frac{n(n-1)}{2}$.‖ These theorems will be very useful in considering what intransitive groups of degree $n-1$ could possibly occur in simply transitive primitive groups of degree n.

When G is k times transitive, its G_1 must be $k-1$ times transitive. When $n=p+\lambda$ or $2p+\lambda$, $\lambda>2$, G cannot be more than λ times transitive without being either the symmetric or the alternating group of degree n.§ It is well known that G includes the alternating group of degree n whenever it involves a transposition or a cyclical substitution of degree three. If two substitutions have only one common element, their commutator is a cyclical substitution of degree three. Hence G cannot include two such substitutions without including the alternating group.

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*Sylow, "Videnskabs-Selskabets Skriften," 1897, No. 9.

†Jordan, "Traité des Substitutions," 1870, p. 284.

‡Miller, Proceedings of the London Mathematical Society, vol. XXVIII, p. 536.

‖Ibid., p. 534.

§Jordan, Bulletin de la Société Mathématique de France, vol. I, p. 41. Cf. Miller, Bulletin of the American Mathematical Society, vol. IV (1898), p. 142.

We observed that G_1 is a maximal subgroup of G. It evidently does not include any self-conjugate subgroup of G besides identity. Conversely, it is easy to prove that if a group of order g contains a maximal subgroup of order g which does not include any self-conjugate subgroup of the entire group, with the exception of identity, then this entire group may be represented as a primitive substitution-group of degree $g \div g_1$ and the number of ways in which it can be represented as such a primitive group may be directly determined from its abstract group properties.[*] We can also determine how many times such a primitive group is transitive from its abstract group properties.[†]

If G_1 is intransitive, every substitution of G that is not contained in G_1 must transform more than $G \div n(n-1)$ substitutions of G_1 into substitutions of G_1, g being the order of G. If it is transitive, every substitution of G that is not included in G_1 must transform just $g \div n(n-1)$ substitutions of G_1 into substitutions of this subgroup. Similar remarks evidently apply to G_1 and its subgroups in case G is more than doubly transitive.

PART II.

Determination of the Groups whose Degree does not exceed Eight.

By means of the methods given in Part I, it is easy to determine all the substitution-groups whose degree does not exceed eight. It is, however, not to be inferred that these methods are sufficient to determine all the groups of large degrees with a reasonable amount of labor. The difficulty of the problem to determine all the possible groups of degree n increases very rapidly with the increase of n, and little progress has been made towards its complete solution. The theorems of Cauchy, Sylow and Frobenius, which relate to the existence of certain subgroups in all the abstract groups of a given order and give some general conditions which the number of these subgroups must satisfy, are probably the most important steps in this direction.

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[*] Dyck, Mathematische Annalen, vol. XXII, p. 90. Cf. Miller, Bulletin of the American Mathematical Society, vol. III, p. 213.

[†] Messenger of Mathematics, 1898, pp. 104-107.

The theorems of Jordan, Bochert and Maillet, which relate to the class of primitive substitution-groups, are also of considerable general importance. A vast number of more special theorems have been published during recent years. Some of these will be very useful in the considerations which follow. In considering the groups of degree n we shall confine our attention to those which actually involve n letters. We shall thus give each substitution-group that does not involve more than eight letters once and only once. The notation explained by Cayley, Quarterly Journal of Mathematics, vol. XXV, will be employed throughout.

§1.—*Determination of the Groups whose Degree does not exceed Five.*

If we have only two letters, it is evident that the only possible permutation is obtained by interchanging the letters. Two such permutations performed in succession bring the letters to their original positions. If the letters are represented by a, b, these operations may be designated by ab and $(ab)^2 = 1$ respectively. They evidently form the only possible group of degree two. This group is designated by (ab).

The symmetric group of degree three is of order 6 and will be represented by (abc) all,[*] while the alternating group of this degree is of order 3 and will be represented by (abc) cyc. There can be no other transitive group of this degree because the order of a transitive group must be divisible by its degree, and there is only one group of degree n whose order is equal to that of the symmetric group of this degree, since this includes all the possible substitutions of degree n.[†] Since the symmetric group of degree n contains only one subgroup (the alternating group) whose order is obtained by dividing the order of the symmetric group by 2,[‡] there can be only one group of degree three and of order 3. From the fact that a system of intransitivity must involve at least two elements, it follows directly that there can be no intransitive group of degree three.

An intransitive group of degree four must contain two transitive constituents of degree two. Hence there are two such groups. The first is obtained by forming the direct product of (ab) and (cd). It is of order 4, and will be represented by $(ab)(cd)$. The second is obtained by making these constituents simply isomorphic. It is represented by $(ab.cd)$. The symmetric group of degree four $(abcd)$ all and the alternating group of this degree $(abcd)$ pos are evidently of orders 24 and 12 respectively. If other transitive groups exist, they must be of order 8 or 4. From Sylow's theorem it follows that all the substitution-groups of order 8 that are found in $(abcd)$ all are conjugate, and that such groups exist. Hence, there is just one transitive group of order 8 and degree four. It is represented by $(abcd)_8$. Since the transitive groups of order 4 must be regular, there is a 1, 1 correspondence between these groups and the abstract groups of order 4;* i. e. there are two transitive groups of degree and order four. They are denoted by $(abcd)_4$ and $(abcd)$ cyc.

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[*] See explanations, p. 335.

[†] Abatti, "Memorie della società italiana delle scienze," t. 10, 1803, p. 385.

[‡] Jordan, "Traité des substitutions" (1870), p. 66.

We have now considered all the possible cases and we have found the seven possible substitution-groups of degree four. Only two of these are simply isomorphic, viz. $(ab)(cd)$ and $(abcd)_4$. Hence the substitution-groups of degree four represent six distinct abstract groups. Since $(abcd)_4$ contains all the substitutions of order 2 and degree four that can be formed with four letters, and is generated by these substitutions, it must be self-conjugate in the alternating and in the symmetric group of degree four. It is well known that the alternating group of every other degree is simple † and that $(abcd)$ all is the only symmetric group that contains more than one self-conjugate subgroup besides identity. Since $(abcd)_4$ is also contained in $(abcd)_8$, it is a self-conjugate subgroup of each of the groups of degree four whose order exceeds 4.

The substitutions of the given group of order 8 may be obtained in various ways. For instance, since there are just n substitutions of degree n that are commutative to each substitution of a regular group involving the same n letters,‡ there are just four substitutions that involve no letter except a, b, c, d, and are commutative to each of the substitutions of $(abcd)$ cyc. These form $(abcd)$ cyc, since this group is Abelian. The total number of substitutions that transform $(abcd)$ cyc into itself and involve no letters except a, b, c, d, must therefore form the required group of order 8. These substitutions are

$$1,\quad abcd,\quad ac.bd,\quad adcb,\quad ac,\quad bd,\quad ab.cd,\quad ad.bc.$$

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* Cf. Netto, "Theory of Substitutions," 1892, p. 110.

† Jordan, "Traité des substitutions," 1870, p. 66.

‡ Jordan, "Traité des substitutions," 1870, p. 60; cf. Capelli, Giornale di Matematiche (1878), p. 40; Dyck, Mathematische Annalen, vol. XX (1882), p. 30.

We arrive at the same substitutions by observing that $(abcd)$ all transforms the substitutions of $(abcd)_4$ according to (abc) all. Hence, one of the conjugate groups of order 8 must correspond to each of the three subgroups of order 2 in (abc) all. A third simple method follows from the fact that the total number of substitutions of degree four that transform $ac \, . \, bd$ into itself is eight, since there are just four such substitutions $(ac)(bd)$ that transform $ac \, . \, bd$ into itself without permuting its two systems of intransitivity.

It is also evident that we arrive at the same group when we construct the largest possible imprimitive group of degree four, since such a group contains just four substitutions that do not permute the two possible systems of imprimitivity of the group and four others that permute these systems. Finally, we may observe that only one of the five possible abstract groups of order 8* contains operators of order 2 that are not self-conjugate, and that all such operators of this group may be made to correspond in some simple isomorphism of the group to itself. Hence, there is only one transitive substitution-group of order 8 and degree four.†

We have now proved the existence of one and only one substitution-group of order 8 and degree four by means of several distinct methods. It is generally possible to prove the existence or non-existence of groups of a given kind by a large number of different methods. We shall, however, usually restrict ourselves to one method. From the preceding it follows that $(abcd)_8$ contains all of the given groups of order 4 as self-conjugate subgroups,‡ and that it contains only one self-conjugate subgroup of order 2. Hence it contains no other self-conjugate subgroup besides identity. As identity is self-conjugate in every group, we shall not always refer to it in enumerating the self-conjugate subgroups of a given group.

Since an intransitive group of degree five must contain one transitive constituent of degree three and one of degree two, there are just three such groups, viz. the direct product of (abc) cyc and (de), the direct product of (abc) all and (de), and the group obtained by establishing a 3, 1 correspondence between (abc) all and (de).§ The orders of these groups are 6, 12, 6 respectively, and they are evidently distinct abstract groups.

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* Cayley, Philosophical Magazine, vol. XVIII, p. 34.

† Dyck, Mathematische Annalen, vol. XXII, p. 90; cf. Miller, Bulletin of the American Mathematical Society, vol. III, 1897, p. 215.

‡ Cf. Netto, "Theory of Substitutions," 1892, p. 81.

§ Since (abc) all has a quotient group of order 2.

Since the divisors of 4 are 1, 2, 4, there are just three groups of degree five that contain one and only one subgroup of order 5.* The orders of these groups are 5, 10, 20. The first is cyclical, the second is semi-metacyclic, and the last is the metacyclic group of degree five. These groups are denoted by $(abcde)$ cyc, $(abcde)_{10}$ and $(abcde)_{20}$ respectively. There cannot be any other transitive group of degree five besides the alternating and the symmetric group, for such a group has to contain six subgroups of order 5 according to Sylow's theorem, and each of these subgroups must be transformed into itself by at least ten substitutions.† Hence the order cannot be less than 60.

§2.—*Determination of the Groups of Degree Six.*

If such a group is intransitive, the degrees of its transitive constituents must be one of the three sets 2, 2, 2 : 3, 3 : 4, 2. The largest group, all of whose transitive constituents are of degree two, is $(ab)(cd)(ef)$, and there is evidently only one such group. Since this contains only four positive substitutions, there is only one positive group of order 4 that contains three systems of intransitivity. As the average number of letters in all the substitutions of such a group is 3, a group of order 4 that involves negative substitutions must contain a transposition. This transposition and the positive substitution involving the other four elements must generate the group of order 4. Hence, there are two and only two groups of order 4 and degree six that contain three systems of intransitivity. There is evidently only one group of order 2 that contains these three systems. We have now found the four possible groups of degree six that involve the systems 2, 2, 2.

Since (abc) cyc and (def) all do not have the same quotient group, there is only one group that contains them as transitive constituents, viz. their direct product (abc) cyc (def) all. The two groups whose transitive constituents are (abc) cyc and (def) cyc are their direct product (abc) cyc (def) cyc and the group obtained by making them simply isomorphic $(abc.def)$ cyc. If the transitive constituents are (abc) all and (def) all, we may form their direct product (abc) all (def) all, their simple isomorphism $(abc.def)$ all, and a 3, 3 correspondence between them $\{(abc)$ all (def) all$\}$ pos. Since (abc) all is a complete group,‡ we obtain only one group by making it simply isomorphic to itself.

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* Cf. Quarterly Journal of Mathematics, vol. XXIX, p. 226.

† See p. 308.

‡ Hölder, Mathematische Annalen, vol. XLVI (1895), p. 325.

Hence, there are just six groups whose transitive constituents are 3, 3. All of them are evidently distinct abstract groups.

If the transitive constituents are of degrees four and two, we obtain five groups by forming the direct product of the transitive groups of degree four and (*ef*). The orders of these five groups [(*abcd*) all (*ef*), (*abcd*) pos (*ef*), $(abcd)_8(ef)$, $(abcd)_4(ef)$, (*abcd*) cyc (*ef*)] are 48, 24, 16, 8, 8 respectively. We have seen that (*abcd*) all and (*abcd*) cyc contain one self-conjugate subgroup which includes one-half of the operators of the group, and that $(abcd)_8$ contains three such subgroups that cannot be transformed into each other while the three subgroups of order 2 in $(abcd)_4$ are transformed into each other by substitutions that transform $(abcd)_4$ into itself. Hence we obtain the following six groups by dimidiation: {(*abcd*) all (*ef*)} pos, {(*abcd*) cyc (*ef*)} pos, $\{(abcd)_4(ef)\}$ dim, and $\{(abcd)_8(ef)\}$ dim. The last symbol includes three groups which are simply isomorphic to $(abcd)_8$. Hence, there are 11 intransitives groups of degree six involving the systems 4, 2.

The imprimitive groups of degree six* have either two or three systems of imprimitivity. We shall first consider those groups which contain two such systems and afterward those which contain three systems without containing also two systems. From the general theory of Part I, it follows directly that each of these groups must contain one of the following five groups:

(*abc*) all (*def*) all, {(*abc*) all (*def*) all} pos, (*abc*.*def*) all,
(*abc*) cyc (*def*) cyc, (*abc*.*def*) cyc,

and that we obtain an imprimitive group whose order is twice the order of the head by adding a substitution of order two (*ad*.*be*.*cf*) to each one of these heads. From theorems III and and V it follows directly that there is only one group for each of the three heads (*abc*) all (*def*) all, (*abc*.*def*) all, (*abc*) cyc (*def*) cyc; and from theorem VI it follows that there are just two groups with the head (*abc*.*def*) cyc. The orders of these five imprimitive groups are evidently 72, 12, 18, 6, 6 respectively. From theorem IV we observe that there are just two such groups that contain {(*abc*) all (*def*) all} pos. It is evident that these seven possible imprimitive groups of degree six, which contain two systems of imprimitivity, represent seven distinct abstract groups.

If an imprimitive group of degree six contains three systems of imprimitivity, it must have one of the following four groups as a head, and its order must be the order of its head multiplied by three or six:

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* Cf. Burnside, "Theory of Groups" (1897), p. 180.

$$1,\quad (ab\,.\,cd\,.\,ef),\quad \{(ab)(cd)(ef)\}\text{ pos},\quad (ab)(cd)(ef).$$

If it has either of the first two heads, its order must be 6 or 12. We observed above that the two possible regular groups of order 6 contain two systems of imprimitivity. It is evident that each of them contains also three systems, but we desire to consider only those groups which contain three systems without also containing two systems. The five possible groups of order 12 are well known,* and only two of them contain subgroups of order 2 that are not self-conjugate. As all these subgroups can be made to correspond in some simple isomorphism of the group to itself, and as they are not maximal, there are just two imprimitive groups of degree six and order 12. We found above the one that contains a self-conjugate subgroup of order 2; the one that is simply isomorphic to $(abcd)$ pos must, therefore, contain the third of the four heads given above, and it must permute its three systems according to the group of order 3.

Since all the operation-groups whose order is less than 64 are known,† we could readily find all the possible imprimitive groups that contain the given heads directly from the simply isomorphic operation-groups. It may, however, be more simple to proceed as follows: From theorem III, Part I, it follows that all the possible imprimitive groups of degree six which involve three systems of imprimitivity, must be contained in the group of order 48 which contains the head $(ab)(cd)(ef)$ and permutes its systems of intransitivity according to the symmetric group of degree three. It remains only to consider the subgroups of order 24 that are contained in this group. Since this group contains only one subgroup of order 12 and is isomorphic to the four-group with respect to this subgroup, it contains just three subgroups of order 24. That each of these three groups contains three systems of imprimitivity follows directly from the fact that the given group of order 48 permutes the systems of intransitivity of the given head according to the symmetric group of degree three. From the same fact we observe that these three groups of order 24 are distinct substitution-groups. The two which involve only the positive substitutions of $(ab)(cd)(ef)$ are simply isomorphic to $(abcd)$ all, while the one which contains $(ab)(cd)(ef)$ is the direct product of the alternating group of degree four and an operator of order 2.

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* Cayley, American Journal of Mathematics, vol. XI, pp. 139-157; cf. Kempe, Philosophical Transactions, vol. CLXXVII, pp. 37-43.

† Miller, Quarterly Journal of Mathematics, vol. XXIX, pp. 243-263.

We have now considered all the possible imprimitive groups of degree six and found that there are seven such groups that contain two systems of imprimitivity, and that there are five that contain three systems without containing also two systems, while there are just three groups that contain both two and three systems, viz. the two regular groups and one of the groups of order 12. The two given groups of order 24 are the only instances where two imprimitive groups of degree six represent the same abstract group.

Mathieu proved* that there is at least one triply transitive group of degree $p+1$ and order $p(p^2-1)$ and one doubly transitive group [the group of the modular equation] of degree $p+1$ and order $\frac{p}{2}(p^2-1)$, p being any prime number. The latter is simple whenever p exceeds 3, and it is formed by the positive substitutions of the former. It will appear that there is no other primitive group of degree six besides the alternating and the symmetric group.

Since each of the intransitive groups of degree five contains a substitution of the form abc, there can be no primitive group of degree six unless its order is divisible by 5. If such a group does not contain the alternating group, its order must be 30, 60 or 120. It could not be of order 30, since every group of this order contains a subgroup of order 15,† and a group of order 15 must be cyclical since $5-1$ is not divisible by 3. The only one of the thirteen possible groups of order 60 that contains six subgroups of order 5 is simply isomorphic to $(abcde)$ pos.‡ As the latter contains only one system of subgroups of order 12 that can be made to correspond, it can be represented in only one way as a transitive group of degree six, and this must therefore be the group mentioned above. If there is a primitive group of order 120, it must contain the given group of order 60 as subgroup, and it must be completely determined by this subgroup, since $(abcde)_{20}$ is completely determined by $(abcde)_{10}$. Hence, we have determined all the possible groups of degree six.

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* Journal de Mathematiques, vol. V (1860), p. 37.

† Frobenius, Berliner Sitzungsberichte, 1895, p. 1043; cf. Hölder, Göttingen Nachrichten, 1895, p. 211.

‡ Burnside, "Theory of Groups of a Finite Order," 1897, p. 107.

§3.—*Determination of the Groups of Degree Seven.*

Since seven is odd, one of the transitive constituents of each intransitive group of this degree must be either of degree three or of degree five. In the former case the other constituent may be either transitive or intransitive. Hence, every intransitive group of degree seven must have one of the following three sets of systems of intransitivity: 3, 4; 3, 2, 2; 5, 2. Since there are five transitive groups of degree four and two of degree three, we can construct just ten intransitive groups by forming the direct product of a transitive group of degree three and one of degree four. $(abcd)$ all is the only group of degree four that has a non-cyclical quotient group of order 6, and $(abcd)$ pos is the only one of these groups that has a quotient group of order 3; hence we obtain just two groups by establishing an α, 1 isomorphism between the transitive groups of degree four and the groups of degree three. The only other quotient group that is represented by the groups of degree three is of order 2, and we have seen that this gives rise to six groups. Hence, there are just eighteen intransitive groups of degree seven whose transitive constituents are of degrees three, four.*

Since there are only two intransitive groups of degree four, we obtain only four groups by forming the direct products of the intransitive groups of degree four and the groups of degree three. The only quotient group which these constituent groups have in common is of order 2. From the fact that $(ab)(cd)$ contains two subgroups of order 2 that cannot be transformed into each other, it follows that we obtain three intransitive groups by dimidiating (abc) all and the intransitive groups of degree four. Hence there are just seven groups of degree seven whose transitive constituents are of degrees 3, 2, 2.

By multiplying (fg) into the five transitive groups of degree five, we obtain the five direct products whose transitive constituents are of degrees five and two. Since three of these transitive groups have one and only one subgroup of half their order, we obtain three additional intransitive groups by dimidiation. Hence, there are eight intransitive groups of degree seven whose transitive constituents are of degrees five and two, and the total number of intransitive groups of degree seven is thirty-three.

It follows directly from Part I that there are just four primitive groups of degree seven which contain only one subgroup of order 7, and that the orders of these groups are 7, 14, 21 and 42 respectively. The substitutions of order 7 that are contained in any other primitive group of this degree must generate a self-conjugate simple group of a composite order. The lowest order of a simple group that contains operators of order 7 is 168, and there is only one simple group of this order.[*] As all the subgroups of order 24 that are found in this simple group can be made to correspond in some simple isomorphism of the group to itself, it can be represented in only one way as a transitive group of degree seven.

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*Cf. Burnside, "Theory of Groups of a Finite Order," 1897, p. 165.

This simple group cannot occur as a self-conjugate subgroup of a larger primitive group of degree seven, since the groups of order 48 and degree six contain a transposition.[†] If other primitive groups of degree seven exist, they must include some other simple group of composite order as a self-conjugate subgroup. The only groups of degree six that could possibly occur in such a simple group are $(abcdef)_{60}$ and $(abcdef)$ pos. The former is impossible because there is no simple group of order 420,[‡] and the latter leads to the alternating group of degree seven. Hence there is one and only one primitive group of each of the orders 7, 14, 21, 42, 168, $\frac{1}{2}\,7!$, $7!$

§4.—*Determination of the Groups of Degree Eight.*

An intransitive group of degree eight must have one of the following five sets of systems of intransitivity: 2, 2, 2, 2; 3, 3, 2; 3, 5; 4, 2, 2; 4, 4; 6, 2. The first of these sets was considered in Part I, where we proved that there are just eight groups of this kind. Those of the second set may be directly obtained from the groups of degree six whose systems of intransitivity are 3, 3. As there are six such groups of degree six, we obtain six groups of degree eight as the direct products of these groups of degree six and a transposition. Two of the groups of degree six, $[(abc.def)$ cyc, (abc) cyc (def) cyc$]$, do not contain a quotient group of order 2; three others, $[(abc.def)$ all, (abc) all (def) cyc, $\{(abc)$ all (def) all$\}$ pos$]$, contain a single group whose order is half the order of the group, while the last, (abc) all (def) all, contains three subgroups of order 18. As two of the last three subgroups are transformed into each other by substitutions that transform (abc) all (def) all into itself, there are just eleven intransitive groups of degree eight which involve the systems 3, 3, 2.

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[*] Hölder, Mathematische Annalen, vol. XL (1892), p. 83.

[†] Cayley, Quarterly Journal of Mathematics, vol. XXV, 1891, p. 81.

[‡] Cole, American Journal of Mathematics, vol. XV (1893), p. 303.

By forming the direct products of the transitive groups of degree five and those of degree three, we obtain ten groups whose systems of intransitivity are 5, 3. As none of the quotient groups of the transitive groups of degree five is of order 3 or 6, we cannot form any groups by establishing an α, 1 isomorphism between the transitive groups of degree five and those of degree three. Hence, it is only necessary to add the three groups obtained by dimidiating some transitive group of degree five and the symmetric group of degree three in order to get the total number (13) of the groups of degree eight whose transitive constituents are of degrees five and three.

If we multiply each of the five transitive groups of degree four into the two intransitive ones, we obtain ten groups whose transitive constituents are of degrees four, two, two. By dimidiating the same groups we obtain eighteen additional groups with the given transitive constituents, since $(ab)(cd)$ has two subgroups of order 2 that cannot be transformed into each other. By establishing an α, 1 isomorphism between the transitive groups of degree four and $(ab)(cd)$, we obtain three additional groups from $(efgh)_8$ and one from $(efgh)_4$. Hence, the total number of these groups whose transitive constituents are 4, 2, 2 is 32. It is evident that the groups obtained by these different methods are distinct.

We obtain fifteen groups by forming the direct products of the transitive groups of degree four into themselves written in a different set of letters. Each of the transitive groups of degree four is transformed into all its simple isomorphisms to itself by the symmetric group of this degree,* with the exception of $(abcd)_8$. Since the group of isomorphism of the last group is simply isomorphic to the group itself, and it has three conjugates in $(abcd)$ all, we obtain just two distinct substitution-groups by making it simply isomorphic to itself. Hence, we obtain six additional groups by establishing a simple isomorphism between two transitive groups of degree four. We proceed to find the remaining groups whose transitive constituents are 4, 4.

The quotient groups of $(abcd)$ all which differ from the entire group are the group of order 2 and the symmetric group of order 6. We can evidently use the latter of these two only in establishing a 4, 4 correspondence between $(abcd)$ all and $(efgh)$ all and thus forming a group of order 96. The former can be used with all the groups that have a quotient-group of half their order, and hence leads to six additional groups. We have now found all the groups whose systems are 4, 4 and which contain the symmetric group as one of their transitive constituents, in addition to those which are obtained by forming the direct product or by establishing a simple isomorphism. We shall next find all the additional groups which have $(abcd)$ pos for one of their transitive constituents, then we shall find those which have $(abcd)_8$ as a transitive constituent without containing either $(abcd)$ all or $(abcd)$ pos, etc.

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* Cf. Hölder, Mathematische Annalen, vol. XLVI (1895), p. 340; Miller, Bulletin of the American Mathematical Society, vol. I (1895), p. 258.

Since $(abcd)$ pos contains only one self-conjugate subgroup besides identity, and this corresponds to a quotient-group of order 3, we obtain only one additional group that contains this group for one of its transitive constituents, viz. the group of order 48 which is formed by establishing a 4, 4 correspondence of the alternating group to itself written in a different set of letters. When one of the transitive constituents is $(abcd)_8$, we have to consider two quotient groups, viz. the four-group and the group of order 2. The former gives rise to five additional groups of order 16* and one of order 8, while the latter leads to six additional groups of order 32† and six of order 16. Hence, there are eighteen groups that contain $(abcd)_8$ as a transitive constituent but do not contain any transitive constituent of a larger order, and are neither the direc- product of two transitive groups of degree four nor the groups obtained by making $(abcd)_8$ simply isomorphic to itself. Since we obtain three additional groups by dimidiating $(abcd)_4$ and $(abcd)$ cyc written in two systems of letters, the total number of the groups of degree eight whose transitive constituents are 4, 4 is 50.

By multiplying each of the sixteen transitive groups of degree six by a transposition we obtain sixteen intransitive groups whose transitive constituents are 6, 2. It remains to examine all the transitive groups of degree six in regard to their subgroups whose orders are obtained by dividing the order of the group by 2 and which cannot be transformed into each other by any substitution that transforms the group into itself. It is well known that each of the groups of order 6 and of order 18 contains only one such subgroup. One of the groups of order 12 contains three such subgroups, while the other does not contain any subgroup of order 6. We have already observed that each of the three transitive groups of order 24 contains only one subgroup of order 12, and that the group of order 48 contains three distinct subgroups of order 24. The group of order 36 which contains substitutions of order 4 contains only one subgroup of order 18, since it is isomorphic to the cyclical group of order 4 with regard to its subgroup of order 9, while the other group of order 36 contains three subgroups of order 18, two of which are conjugate in the group of order 72. Since the group of order 72 is isomorphic to $(abcd)_8$ with respect to its subgroup of order 9, it contains just three subgroups of order 36—the two transitive groups just mentioned and the direct product of two symmetric groups of degree three. Hence, we obtain eighteen groups by dimidiating the imprimitive groups of degree six and *(gh)*.

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* Cole, Bulletin of the New York Mathematical Society, vol. II (1893), p. 187; cf. Miller, ibid., vol. III (1894), p. 168.

† Cayley, Quarterly Journal of Mathematics, vol. XXV (1891), p. 147.

Since the four primitive groups of degree six (regarded as abstract groups) are the symmetric and the alternating groups of degrees five and six, we obtain only two groups by dimidiating a primitive group of degree six and *(gh)*. Hence, there are thirty-six groups of degree eight whose transitive constituents are of degrees six, two. The total number of intransitive groups of this degree is therefore $8 + 11 + 13 + 32 + 50 + 36 = 150$. It remains to determine the transitive groups.

The imprimitive groups of degree eight contain either two or four systems of imprimitivity. We shall first consider those which contain two such systems and afterwards those which contain four systems without containing also two systems. The following twenty-three groups may be used as heads: *(abcd)* all *(efgh)* all, $\{(abcd)$ all $(efgh)$ all$\}$ pos, $\{(abcd)$ all $(efgh)$ all$\}_{4,4}$, *(abcd . efgh)* all, *(abcd)* pos *(efgh)* pos, $\{(abcd)$ pos $(efgh)$ pos$\}_{4,4}$, *(abcd . efgh)* pos, $(abcd)_8\,(efgh)_8$, $\{(abcd)_8\,(efgh)_8\}$ dim (three groups), $\{(abcd)_8\,(efgh)_8\}_{2,2}$ (four groups), $(abcd\,.\,efgh)_8$ (two groups), $(abcd)_4\,(efgh)_4$, $\{(abcd)_4\,(efgh)_4\}$ dim, $(abcd\,.\,efgh)_4$, *(abcd)* cyc *(efgh)* cyc, $\{(abcd)$ cyc $(efgh)$ cyc$\}$ pos, *(abcd . efgh)* cyc.

We proved in Part I that there are just five distinct imprimitive groups of degree eight in which the transitive constituents of the head are the symmetric groups of degree four, and that there are five such groups in which the transitive constituents of the head are the alternating groups of degree four. Hence, the first seven of the given twenty-three heads give rise to ten imprimitive groups of degree eight. Three of the remaining heads are the direct products of two transitive groups, and, hence, each gives rise to only one imprimitive group. It is evident that these three groups are distinct from each other and from those which involve the other heads.

From theorem IV it follows that each of the three heads represented by $\{(abcd)_8\,(efgh)_8\}$ dim gives rise to two groups, and that these six groups of order 64 are distinct. From the same theorem we observe that the group represented by $\{(abcd)_8\,(efgh)\}_{2,\,2}$ in which the identical divisions correspond, is the head of four distinct groups of order 32, and also that each of the other three heads represented by this notation leads to two conjugate groups. Two of the last three groups involve substitutions of degree six, and must therefore be distinct from the other groups of order 32 which have been given above; the third contains a subgroup of order 16, which is conjugate to the $\{(abcd)_8\,(efgh)_8\}_{2,\,2}$ in which the identical divisions correspond, and therefore is conjugate to one of the four groups containing this head. Hence, there are just six distinct groups that contain one of the four heads represented by $\{(abcd)_8\,(efgh)_8\}_{2,\,2}$.[*]

As the remaining heads lead to groups of order 8 or 16, it may be easiest to obtain them from their simply isomorphic abstract groups. Since each abstract group can be represented in just one way as a regular group, there are five regular groups of degree eight, and their substitutions[†] can be directly obtained from the simply isomorphic abstract groups. It remains to consider the groups of order 16 which can be represented as transitive groups of degree eight, and to determine in how many ways each group can be so represented. Since an Abelian or a Hamiltonian group cannot be transitive unless it is regular, we may confine our attention to the eight groups of order 16 that are neither Abelian nor Hamiltonian. A list of these groups, written out in full, is given in Quarterly Journal of Mathematics, vol. XXVIII, pp. 269–273.

It is evident that the groups Nos. 6 and 11 in this list do not contain any subgroup of order 2 that is not self-conjugate, and that each of the remaining six groups contains such subgroups. In at least five of these groups all of these subgroups of order 2 can be made to correspond in some simple isomorphism of the group to itself, since they do not occur in one of the Abelian subgroups of order 8; hence, each of these groups can be represented in only one way as a transitive substitution group of degree eight. No. 9 contains three Abelian subgroups of order 8, each containing two subgroups of order 2 that are not self-conjugate in the entire group. From the group of isomorphisms of this group,[‡] it follows directly that all of these subgroups of order 2 correspond in some simple isomorphism of the group to itself. Hence, it also can be represented in only one way as a transitive substitution group of degree eight. From this we see that there are just six transitive substitution-groups of degree eight and order 16. These must be imprimitive, since a group of order p^α, p being any prime number and $\alpha > 1$, cannot be primitive. They must contain two systems of imprimitivity, since they could not be isomorphic to a primitive group of degree four.

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[*] Cole, Bulletin of the New York Mathematical Society, vol. II (1893), p. 188.

[†] Cayley, Quarterly Journal of Mathematics, vol. XXV (1891), p. 143.

[‡] Miller, Quarterly Journal of Mathematics, vol. XXVIII, p. 253.

We have now considered all the possible cases and found that there are just thirty-six groups of degree eight that contain two systems of imprimitivity. It remains to determine those which contain four systems without also containing two systems. Since such groups must permute their systems according to $(abcd)$ pos or $(abcd)$ all, the systems of intransitivity of the heads must permit the permutations represented by these groups. Hence, we need to consider only the following heads:

$$1, \quad (ab\,.\,cd\,.\,ef\,.\,gh), \quad \{(ab)(cd)(ef)(gh)\} \text{ pos}, \quad (ab)(cd)(ef)(gh).$$

The first one of these heads does not lead to an additional group, since $(abcd)$ all can be represented in only one way as a transitive substitution-group of degree eight, having only one system of conjugate subgroups of order 3. It is evident that one such representation is possible with the head $(abcd\,.\,efgh)$ pos. The second of these heads gives rise to groups of order 24 and 48 which contain non-maximal subgroups of orders 3 and 6 respectively that are not self-conjugate nor contain any self-conjugate subgroup of the entire group. If a transitive group of order 24 and degree eight contains a subgroup of order 12, this must be intransitive and the group must contain two systems of imprimitivity. Since there is only one group of order 24 that does not contain a subgroup of order 12* and its subgroups of order 3, are non-maximal and conjugate by Sylow's theorem, there is only one imprimitive group of order 24 and degree eight that does not have two systems of imprimitivity.

The groups of order 48 that can be represented as transitive groups of degree eight, must clearly be contained among the eight groups of this order that contain four subgroups of order 3.† Four of these eight groups contain a self-conjugate subgroup of order 12, and hence they must contain two systems of imprimitivity if they can be represented as transitive groups of degree eight. In fact, only one of these four groups (G_{47}) can be represented as a transitive group of degree eight. It is also easily seen that only one of the remaining four groups (G_{51}) contains non-maximal subgroups of order 6 that do not include any operator except identity that is self-conjugate, and that all such subgroups may be made to correspond in some simple isomorphism of G_{51} to itself. Hence, there is only one imprimitive group of order 48 that contains four systems of imprimitivity without containing also two systems. That there are only two groups with the head $(ab)(cd)(ef)(gh)$ and three with the head $\{(ab)(cd)(ef)(gh)\}$ pos following directly from the general theory. Hence there are just seven groups of degree eight that contain four systems of imprimitivity without containing also two such systems. The total number of imprimitive groups of this degree is therefore forty-three.

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* Miller, Quarterly Journal of Mathematics, vol. XXVIII, p. 274.

† Ibid., vol. XXX, p. 258.

Each of the solvable primitive groups of degree eight must contain as a self-conjugate subgroup the regular group of order eight which contains no operator whose order exceeds 2. Since the group of isomorphisms of this regular group is $(abcdefg)_{168}$,[*] these primitive groups must correspond to subgroups of this group of order 168. It is evident that a doubly transitive group of order 56 corresponds to the subgroup of order 7, and that a doubly transitive group of order 168 corresponds to the subgroup of order 21. Since all of the subgroups of orders 7 and 21 are conjugate in $(abcdefg)_{168}$, there is only one solvable primitive group of each of the given orders.

If there were any other solvable primitive group of degree eight, it would have to be simply transitive, and its maximal subgroup of degree seven would have to be a subgroup of $(abcdefg)_{168}$[†] and contain the systems 3, 4.[‡] As this intransitive group could not contain any substitution whose degree is less than four nor a transitive group whose degree is less than six, it cannot be constructed. Hence, there are only two solvable primitive groups of degree eight.

From what precedes, it follows that the largest group of degree eight which contains the given regular group as a self-conjugate subgroup is triply transitive and of order 1344. The two Mathieu groups of orders $p(p^2-1)$, $\frac{p}{2}(p^2-1)$

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[*] Moore, Bulletin of the American Mathematical Society, vol. I (1894), p. 61.

[†] Ibid., vol. V (1899), p. 250.

[‡] Jordan, "Traité des substitutions" (1870), p. 284.

and degree $p+1$ are of orders 336 and 168 respectively. Since the latter is simple, it is distinct from the solvable primitive group of this order which we determined above.* If we add to these groups the alternating and the symmetric, we have seven primitive groups of degree eight, two being solvable and five insolvable. We proceed to prove that there is no other primitive group of this degree.

If a primitive group of degree eight were simply transitive, its maximal subgroup of degree seven would have transitive constituents of degrees three and four. As it could not contain any transitive subgroup whose degree is less than eight† nor any substitution whose degree is less than four, such a group cannot be constructed.‡ Hence, it is only necessary to consider groups of orders 56, 112, 168, 336 and 1344. As we have considered all the possible solvable groups, we may suppose that the groups under consideration contain a composite factor of composition. Hence, we do not need to consider the first two of the given orders. We proved in Part I that there cannot be more than one group of degree eight and order 1344. We may, therefore, restrict ourselves to groups of orders 168 and 336.

Since there is only one simple group of order 168,§ and this group contains only one system of conjugate subgroups of order 21, it can be represented in only one way as a transitive group of degree eight. An insolvable group of order 336 must contain the given simple group as self-conjugate subgroup, and is completely determined by it since $(abcdefg)_{42}$ is completely determined by $(abcdefg)_{21}$. Hence the proof is complete. We have now examined all the possible cases and found that there are just two hundred substitution-groups of degree eight; one hundred and fifty of these are intransitive, forty-three are imprimitive and seven are primitive.

PART III.

List of all the Substitution-Groups whose Degree is less than Nine.

In the following list the groups which are simply isomorphic to groups which precede them, are represented by Roman letters, and the others are represented by Greek letters. All these substitution-groups, which are represented by Greek letters, are therefore distinct as abstract or operation-groups; and they include all the abstract groups which can be represented by substitution-groups whose degree does not exceed eight.

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*Cf. Hölder, Mathematische Annalen, vol. XL, p. 75.

†Cf. Netto, "Theory of Substitutions" (1892), p. 95.

‡Cf. Miller, Proceedings of the London Mathematical Society, vol. XXVIII, p. 534.

§Burnside, "Theory of Groups of a Finite Order," 1897, p. 208.

The number which follows a group represented by Roman letters indicates the degree of the first substitution-group (represented by Greek letters) to which it is simply isomorphic, and the subscript of this number indicates which of the groups of the given degree and order is meant. If there is only one such group, this subscript is omitted. E. g. the 3 after the second and third groups of degree six and order 6 indicates that these two groups are simply isomorphic to the group of degree three and order 6, while the 6_1 after the third group of degree six and order 8 indicates that this is simply isomorphic to the first group of this degree and order.

Each of the insolvable groups is followed by the abbreviation *ins.* It may be observed that only 28 of the 295 substitution-groups whose degree does not exceed eight are insolvable. The number of abstract groups represented by these 295 substitution-groups is 137, i. e. somewhat less than one-half of the total number. The number of abstract insolvable groups is 20; five of these are simple groups of composite order, viz. one group of each of the order 60, 168, 360, 2520 and 20160.

LIST.

Order.	No.	Notation.	
			Degree Two.
2	1	$(\alpha\beta)$.	
			Degree Three.
3	1	$(\alpha\beta\gamma)$ cyc,	
6	1	$(\alpha\beta\gamma)$ all.	
			Degree Four.
2	1	$(ab.cd)$,	2
4	1	$(\alpha\beta)(\gamma\delta)$,	
	2	$(abcd)_4$,	4_1
	3	$(\alpha\beta\gamma\delta)$ cyc,	
8	1	$(\alpha\beta\gamma\delta)_8$,	
12	1	$(\alpha\beta\gamma\delta)$ pos,	
24	1	$(\alpha\beta\gamma\delta)$ all.	
Total,	7		

Degree Five.

Order.	No.	Notation.	
5	1	$(\alpha\beta\gamma\delta\varepsilon)$ cyc,	
6	1	$(\alpha\beta\gamma)$ cyc $(\delta\varepsilon)$	
	2	$\{(abc)$ all $(de)\}$ pos,	3
10	1	$(\alpha\beta\gamma\delta\varepsilon)_{10}$,	
12	1	$(\alpha\beta\gamma)$ all $(\delta\varepsilon)$,	
20	1	$(\alpha\beta\gamma\delta\varepsilon)_{20}$,	
60	1	$(\alpha\beta\gamma\delta\varepsilon)$ pos,	ins.
120	1	$(\alpha\beta\gamma\delta\varepsilon)$ all,	ins.
Total,	8		

Degree Six.

Order.	No.	Notation.	
2	1	$(ab\,.\,cd\,.\,ef)$,	2
3	1	$(abc\,.\,def)$ cyc,	3
4	1	$(ab\,.\,cd)(ef)$,	4_1
	2	$\{(ab)(cd)(ef)\}$ pos,	“
	3	$\{(abcd)_4(ef)\}$ dim,	“
	4	$\{(abcd)$ cyc $(ef)\}$ pos,	4_3
6	1	$(abcdef)$ cyc,	5
	2	$(abc\,.\,def)$ all,	3
	3	$(abcdef)_6$,	“
8	1	$(\alpha\beta)(\gamma\delta)(\varepsilon\zeta)$,	
	2	$(\alpha\beta\gamma\delta)$ cyc $(\varepsilon\zeta)$,	
	3	$(abcd)_4(ef)$,	6_1
	4	$\{(abcd)_8$ com $(ef)\}$ dim,	4
	5	$\{(abcd)_8$ cyc (ef) dim,	“
	6	$\{(abcd)_8$ pos $(ef)\}$ dim,	“
9	1	$(\alpha\beta\gamma)$ cyc $(\delta\varepsilon\zeta)$ cyc,	
12	1	$(abcdef)_{12}$,	5
	2	$(abcdef)_{12_2}$,	4
16	1	$(\alpha\beta\gamma\delta)_8(\varepsilon\zeta)$,	
18	1	$(\alpha\beta\gamma)$ all $(\delta\varepsilon\zeta)$ cyc,	
	2	$\{(\alpha\beta\gamma)$ all $(\delta\varepsilon\zeta)$ all$\}$ pos,	
	3	$(abcdef)_{18}$,	6_1

Order.	No.	Notation.		
24	1	$(\alpha\beta\gamma\delta)$ pos $(\varepsilon\zeta)$,		
	2	$\{(abcd)$ all $(ef)\}$ pos,		4
	3	$(\pm\, abcdef)_{24}$,		“
	4	$(+\, abcdef)_{24}$,		“
	5	$(abcdef)_{24_5}$,		6_1
36	1	$(\alpha\beta\gamma)$ all $(\delta\varepsilon\zeta)$ all,		
	2	$(\alpha\beta\gamma\delta\varepsilon\zeta)_{36}$,		
	3	$(abcdef)_{36_3}$,		6_1
48	1	$(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta)$,		
	2	$(abcdef)_{48}$,		6_1
60	1	$(abcdef)_{60}$,	ins.	5
72	1	$(\alpha\beta\gamma\delta\varepsilon\zeta)_{72}$,		
120	1	$(abcdef)_{120}$,	ins.	5
360	1	$(\alpha\beta\gamma\delta\varepsilon\zeta)$ pos,	“	
720	1	$(\alpha\beta\gamma\delta\varepsilon\zeta)$ all,	“	
Total,	37			

Degree Seven.

6	1	$(ac\,.\,bd)(efg)$ cyc,	5
	2	$\{(ac\,.\,bd)(efg)$ all$\}$ dim,	3
7	1	$(\alpha\beta\gamma\delta\varepsilon\zeta\eta)$ cyc,	
10	1	$(\alpha\beta\gamma\delta\varepsilon)$ cyc $(\zeta\eta)$,	
	2	$\{(abcde)_{10}(fg)\}$ dim,	5
12	1	$(ac\,.\,bd)(efg)$ all,	“
	2	$(\alpha\beta)(\gamma\delta)(\varepsilon\zeta\eta)$ cyc,	
	3	$(\alpha\beta\gamma\delta)$ cyc $(\varepsilon\zeta\eta)$ cyc,	
	4	$(abcd)_4(efg)$ cyc,	7_2
	5	$\{(ab)(cd)(efg)$ all$\}$ pos,	5
	6	$\{(\alpha\beta\gamma\delta)$ cyc $(\varepsilon\zeta\eta)$ all$\}$ pos,	
	7	$\{(abcd)_4(efg)$ all$\}$ dim,	5
	8	$\{(abc)$ all $(de\}$ pos (fg),	“
	9	$\{(abcd)$ pos (efg) cyc$\}$ tris,	4
14	1	$(\alpha\beta\gamma\delta\varepsilon\zeta\eta)_{14}$,	
20	1	$(\alpha\beta\gamma\delta\varepsilon)_{10}(\zeta\eta)$,	

Order.	No.	Notation.		
	2	$\{(abcde)_{20}(fg)\}$ pos,		5
21	1	$(\alpha\beta\gamma\delta\varepsilon\zeta\eta)_{21}$,		
24	1	$(\alpha\beta)(\gamma\delta)(\varepsilon\zeta\eta)$ all,		
	2	$(\alpha\beta\gamma\delta)$ cyc $(\varepsilon\zeta\eta)$ all,		
	3	$(abcd)_4(efg)$ all,		7_1
	4	$(\alpha\beta\gamma\delta)_8(\varepsilon\zeta\eta)$ cyc,		
	5	$\{(\alpha\beta\gamma\delta)_8$ com $(\varepsilon\zeta\eta)$ all$\}$ dim,		
	6	$\{(\alpha\beta\gamma\delta)_8$ cyc $(\varepsilon\zeta\eta)$ all$\}$ dim,		
	7	$\{(abcd)_8$ pos (efg) all$\}$ dim,		7_5
	8	$\{(abcd)\text{ all }(efg)\text{ all}\}_{4,1}$,		4
36	1	$(\alpha\beta\gamma\delta)$ pos $(\varepsilon\zeta\eta)$ cyc,		
40	1	$(\alpha\beta\gamma\delta\varepsilon)_{20}(\zeta\eta)$,		
42	1	$(\alpha\beta\gamma\delta\varepsilon\zeta\eta)_{42}$,		
48	1	$(\alpha\beta\gamma\delta)_8(\varepsilon\zeta\eta)$ all,		
72	1	$(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta)$ cyc,		
	2	$(\alpha\beta\gamma\delta)$ pos $(\varepsilon\zeta\eta)$ all,		
	3	$\{(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta)$ all$\}$ pos,		
120	1	$(\alpha\beta\gamma\delta\varepsilon)$ pos $(\zeta\eta)$,	ins.	
	2	$\{(abcde)$ all $(fg)\}$ pos,	ins.	5
144	1	$(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta)$ all,		
168	1	$(\alpha\beta\gamma\delta\varepsilon\zeta\eta)_{168}$,	ins.	
240	1	$(\alpha\beta\gamma\delta\varepsilon)$ all $(\zeta\eta)$,	ins.	
2520	1	$(\alpha\beta\gamma\delta\varepsilon\zeta\eta)$ pos,	ins.	
5040	1	$(\alpha\beta\gamma\delta\varepsilon\zeta\eta)$ all,	ins.	
Total,	40			

Degree Eight.

2	1	$(ab\,.\,cd\,.\,ef\,.\,gh)$,	2
4	1	$(ab\,.\,cd\,.\,ef)(gh)$,	4_1
	2	$(ab\,.\,cd)(ef\,.\,gh)$,	"
	3	$\{(ab)(cd)(ef\,.\,gh)\}$ dim,	"
	4	$\{(abcd)$ cyc $(ef\,.\,gh)\}$ dim,	4_3
	5	$\{(abcd)_4(ef\,.\,gh)\}$ dim,	4_1
	6	$(abcd\,.\,efgh)$ cyc,	4_3

Order.	No.	Notation.	
	7	$\{(abcd)_4(ef)(gh)\}_{1,1}$,	4_1
	8	$(abcd.efgh)_4$,	"
6	1	$(abc.def)$ cyc (gh),	5
	2	$\{(abcdef)$ cyc $(gh)\}$ pos,	"
	3	$\{(abc.def)$ all $(gh)\}$ dim,	3
	4	$\{(abcdef)_6(gh)$ dim,	"
8	1	$(ab.cd)(ef)(gh)$,	6_1
	2	$\{(ab)(cd)(ef)\}$ pos (gh),	"
	3	$\{(abcd)$ cyc $(ef)\}$ pos (gh),	6_2
	4	$\{(abcd)_4(ef)\}$ dim (gh),	6_1
	5	$(abcd)$ cyc $(ef.gh)$,	6_2
	6	$(abcd)_4(ef.gh)$,	6_1
	7	$1,\ abcd,\ ac.bd,\ adcb + ef.gh(ac,\ bd,\ ab.cd,\ ad.bc)$,	4
	8	$1,\ ab.cd,\ ac.bd,\ ad.bc + ef.gh(ac,\ bd,\ abcd,\ adcb)$,	"
	9	$1,\ ac,\ bd,\ ac.bd + ef.gh(abcd,\ adcb,\ ab.cd,\ ad.bc)$,	"
	10	$\{(ab)(cd)(ef)(gh)\}$ pos,	6_1
	11	$\{(abcd)$ cyc $(efgh)$ cyc$\}_{2,2}$,	6_2
	12	$\{(abcd)_4(efgh)_4\}_{2,2}$,	6_1
	13	$\{(ac)(bd)(efgh)$ cyc$\}$ pos,	6_2
	14	$\{(ac)(bd)(efgh)_4\}$ dim,	6_1
	15	$\{(abcd)_4(efgh)$ cyc$\}$ dim,	6_2
	16	$(abcd.efgh)_{8_1}$,	4
	17	$\{(abcd)_8(ef)(gh)\}_{2,1}$,	"
	18	$\{(abcd)_8(ef)(gh)\}_{2,1}$,	"
	19	$\{(abcd)_8(ef)(gh)\}_{2,1}$,	"
	20	$(abcd.efgh)\ 8_2$,	"
	21	$\{(abcd)_8(efgh)_4\}_{2,1}$,	"
	22	$(\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta)$ cyc,	
	23	$A\,(abcdefgh)_8$,	6_1
	24	$B\,(abcdefgh)_8$,	4
	25	$C\,(abcdefgh)_8$,	6_2
	26	$D\,(\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta)_8$,	
12	1	$(abcdef)$ cyc (gh),	7_2
	2	$(abc.def)$ all (gh),	5

Order.	No.	Notation.		
	3	$(abcdef)_6(gh)$,	5	
	4	$\{(abcdef)_{12}(gh)\}$ dim,	"	
	5	$\{(abcdef)_{12}(gh)\}$ pos,	"	
	6	$\{(abcdef)_{12}(gh)\}$ dim$_2$,	"	
	7	$(abcd\,.\,efgh)$ pos,	4	
15	1	$(\alpha\beta\gamma\delta\varepsilon)$ cyc $(\zeta\eta\theta)$ cyc,		
16	1	$(\alpha\beta)(\gamma\delta)(\varepsilon\zeta)(\eta\theta)$,		
	2	$(\alpha\beta\gamma\delta)$ cyc $(\varepsilon\zeta)(\eta\theta)$,		
	3	$(abcd)_4(ef)(gh)$,	8_1	
	4–9	$\{(ab)(cd)(efgh)_8\}$ dim,	6	
	10	$(\alpha\beta\gamma\delta)$ cyc $(\varepsilon\zeta\eta\theta)$ cyc,		
	11	$(abcd)_4(efgh)$ cyc,	8_2	
	12	$(abcd)_4(efgh)_4$,	8_1	
	13	$(abcd)_8(ef\,.\,gh)$,	6	
	14–16	$\{(abcd)_4(efgh)_8\}$ dim,	"	
	17, 18	$\{(\alpha\beta\gamma\delta)$ cyc $(\varepsilon\zeta\eta\theta)_8\}$ dim$_1$,		
	19	$\{(\alpha\beta\gamma\delta)$ cyc $(\varepsilon\zeta\eta\theta)_8\}$ dim$_2$,		
	20, 21	$\{(abcd)_8(efgh)_8\}_{2,\,2}$,	6	
	22–24	$\{(abcd)_8(efgh)_8\}_{2,\,2}$,	8_{17}	
	25	$\{(\alpha\beta\gamma\delta)$ cyc $(\varepsilon\zeta\eta\theta)$ cyc$\}$ pos $(\alpha\varepsilon\,.\,\beta\zeta\,.\,\gamma\eta\,.\,\delta\theta)$,		
	26	$\{(\alpha\beta\gamma\delta)$ cyc $(\varepsilon\zeta\eta\theta)$ cyc$\}$ pos $(\alpha\zeta\beta\eta\gamma\theta\delta\varepsilon)$,		
	27	$\{(abcd)_4(efgh)_4\}$ dim $(ae\,.\,bf\,.\,cg\,.\,dh)$,		6
	28	$\{(abcd)_4(efgh)_4\}$ dim $(afbe\,.\,chdg)$,		8_{17}
	29	$(\alpha\beta\gamma\delta\,.\,\varepsilon\zeta\eta\theta)_{8_2}(\alpha\varepsilon\beta\zeta\gamma\eta\delta\theta)$,		
	30	$(\alpha\beta\gamma\delta)\,.\,\varepsilon\zeta\eta\theta_{8_2}(\alpha\zeta\beta\eta\gamma\theta\delta\varepsilon)$,		
18	1	$(\alpha\beta\gamma)$ cyc $(\delta\varepsilon\zeta)$ cyc $(\eta\theta)$,		
	2	$\{(abc)$ all $(de)\}$ pos (fgh) cyc,		6_1
	3	$\{(abc)$ all (def) all$\}$ pos (gh) dim,		6_2
	4	$\{(abcdef)_{18}(gh)\}$ pos,		6_1
24	1	$(abcdef)_{12}(gh)$,		7_1
	2	$(abcd)$ pos $(ef\,.\,gh)$,		6_1
	3	$\{(abcd)$ all $(ef\,.\,gh)\}$ dim,		4
	4	$(abcd\,.\,efgh)$ all,		"
	5	$\{(+\,abcdef)_{24}(gh)\}$ dim,		"

Order.	No.	Notation.	
	6	$\{(\pm abcdef)_{24}(gh)\}$ pos,	4
	7	$\{(abcdef)_{24_5}(gh)\}$ pos,	6_1
	8	$(abcdef)_{12_2}(gh)$,	"
	9	$(abcd\,.\,efgh)$ pos $(ae\,.\,bf\,.\,cg\,.\,dh)$,	"
	10	$(\alpha\delta\varepsilon\,.\,\beta\gamma\zeta)(\alpha\zeta\beta\varepsilon\,.\,\gamma\theta\delta\eta)$,	
	11	$(abcd\,.\,efgh)$ pos $(af\,.\,be\,.\,cg\,.\,dh)$,	4
30	1	$(\alpha\beta\gamma\delta\varepsilon)$ cyc $(\zeta\eta\theta)$ all,	
	2	$(\alpha\beta\gamma\delta\varepsilon)_{10}(\zeta\eta\theta)$ cyc,	
	3	$\{(\alpha\beta\gamma\delta\varepsilon)_{10}(\zeta\eta\theta)$ all$\}$ dim,	
32	1	$(\alpha\beta\gamma\delta)_8(\varepsilon\zeta)(\eta\theta)$,	
	2	$(\alpha\beta\gamma\delta)_8(\varepsilon\zeta\eta\theta)$ cyc,	
	3	$(abcd)_8(efgh)_4$,	8_1
	4	$L\,\{(\alpha\beta\gamma\delta)_8(\varepsilon\zeta\eta\theta)_8\}$ dim,	
	5	$M\{(\alpha\beta\gamma\delta)_8(\varepsilon\zeta\eta\theta)_8\}$ dim,	
	6	$N\{(abcd)_8(efgh)_8\}$ dim,	8_4
	7	$P\,\{(\alpha\beta\gamma\delta)_8(\varepsilon\zeta\eta\theta)_8\}$ dim,	
	8	$Q\,\{(abcd)_8(efgh)_8\}$ dim,	8_4
	9	$R\,\{(abcd)_8(efgh)_8\}$ dim,	8_7
	10	$(\alpha\beta\gamma\delta)$ cyc) $(\varepsilon\zeta\eta\theta)$ cyc $(\alpha\varepsilon\,.\,\beta\zeta\,.\,\gamma\eta\,.\,\delta\theta)$,	
	11	$(abcd)_4(efgh)_4(ae\,.\,bf\,.\,cg\,.\,dh)$,	8_4
	12	$\{(\alpha\beta\gamma\delta)_8(\varepsilon\zeta\eta\theta)_8\}_{2,\,2}(\alpha\varepsilon\,.\,\beta\zeta\,.\,\gamma\eta\,.\,\delta\theta)$,	
	13	$\{(\alpha\beta\gamma\delta)_8(\varepsilon\zeta\eta\theta)_8\}_{2,\,2}(\alpha\varepsilon\beta\zeta\,.\,\gamma\eta\delta\theta)$,	
	14	$\{(abcd)_8(efgh)_8\}_{2,\,2}(aecg\,.\,bf\,.\,dh)$,	8_{13}
	15	$\{(\alpha\beta\gamma\delta)_8(\varepsilon\zeta\eta\theta)_8\}_{2,\,2}(\alpha\varepsilon\beta\zeta\gamma\eta\delta\theta)$,	
	16	$A\,\{(\alpha\beta\gamma\delta)_8(\varepsilon\zeta\eta\theta)_8\}_{2,\,2}(\alpha\varepsilon\,.\,\beta\zeta\,.\,\gamma\eta\,.\,\delta\theta)$,	
	17	$B\,\{(abcd)_8(efgh)_8\}_{2,\,2}(ae\,.\,bf\,.\,cg\,.\,dh)$,	8_{13}
36	1	$\{(abc)$ all $(de)\}$ pos (fgh) all,	6_1
	2	$\{(\alpha\beta\gamma)$ all $(\delta\varepsilon\zeta)$ all$\}$ pos $(\eta\theta)$,	
	3	$\{(abc)$ all (def) all $(gh)\}$ pos,	6_1
	4	$(\alpha\beta\gamma\delta\varepsilon\zeta)_{18}(\eta\theta)$,	
	5	$\{(abcdef)_{36}(gh)\}$ dim,	6_2
	6	$\{(abcdef)_{36_3}(gh)\}$ pos,	6_1
	7	$\{(abcdef)_{36_3}(gh)\}$ dim,	6_1
	8	(abc) all (def) cyc (gh),	8_4

Order.	No.	Notation.	
48	1	$(abcd)$ all $(ef\,.\,gh)$,	6_1
	2	$(\alpha\beta\gamma\delta)$ pos $(\varepsilon\zeta)(\eta\theta)$,	
	3	$(\alpha\beta\gamma\delta)$ pos $(\varepsilon\zeta\eta\theta)$ cyc,	
	4	$(abcd)$ pos $(efgh)_4$,	8_2
	5	$\{(abcd)$ all $(ef)\}$ pos (gh),	6_1
	6	$(\pm\, abcdef)_{24}\,(gh)$,	ʻʻ
	7	$(+\, abcdef)_{24}\,(gh)$,	ʻʻ
	8	$\{(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta\theta)$ cyc$\}$ dim,	
	9	$\{(abcd)$ all $(ef)(gh)\}$ dim,	6_1
	10	$\{(abcd)$ all $(efgh)_4\}$ dim,	ʻʻ
	11–13	$\{(abcdef)_{48}\,(gh)\}$ dim,	ʻʻ
	14	$\{(\alpha\beta\gamma\delta)$ pos $(\varepsilon\zeta\eta\theta)$ pos$\}_{4,\,4}$,	
	15	$(abcdef)_{24_5}\,(gh)$,	8_2
	16	$(abcd\,.\,efgh)$ all $(ae\,.\,bf\,.\,cg\,.\,dh)$,	6_1
	17	$(\alpha\delta\varepsilon\,.\,\beta\gamma\zeta)(\alpha\gamma\zeta\theta\beta\delta\varepsilon\eta)$,	
56	1	$A\,(\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta)_8\,(\beta\gamma\varepsilon\delta\eta\theta\zeta)$ cyc,	
60	1	$(\alpha\beta\gamma\delta\varepsilon)_{10}\,(\zeta\eta\theta)$ all,	
	2	$(\alpha\beta\gamma\delta\varepsilon)_{20}\,(\zeta\eta\theta)$ cyc,	
	3	$\{(\alpha\beta\gamma\delta\varepsilon)_{20}\,(\zeta\eta\theta)$ all$\}$ pos,	
64	1	$(\alpha\beta\gamma\delta)_8\,(\varepsilon\zeta\eta\theta)_8$,	
	2	$M\{(\alpha\beta\gamma\delta)_8\,(\varepsilon\zeta\eta\theta)_8\}$ dim $(\alpha\varepsilon\,.\,\beta\zeta\,.\,\gamma\eta\,.\,\delta\theta)$,	
	3	$N\{(\alpha\beta\gamma\delta)_8\,(\varepsilon\zeta\eta\theta)_8\}$ dim $(\alpha\varepsilon\,.\,\beta\zeta\,.\,\gamma\eta\,.\,\delta\theta)$,	
	4	$\{(\alpha\beta\gamma\delta)_8\,(\varepsilon\zeta\eta\theta)_8$ cyc$\}$ dim $(\alpha\varepsilon\gamma\eta\,.\,\beta\zeta\,.\,\delta\theta)$,	
	5	$\{(\alpha\beta\gamma\delta)_8\,(\varepsilon\zeta\eta\theta)_8$ pos$\}$ dim $(\alpha\varepsilon\beta\zeta\gamma\eta\delta\theta)$,	
	6	$\{(abcd)_8\,(efgh)_8$ com$\}$ dim $(ae\,.\,bf\,.\,cg\,.\,dh)$,	8_3
	7	$\{(abcd)_8\,(efgh)_8$ com$\}$ dim $(aebfcgdh)$,	8_5
72	1	$(\alpha\beta\gamma)$ all $(\delta\varepsilon\zeta)$ all $(\eta\theta)$,	
	2	$(\alpha\beta\gamma\delta\varepsilon\zeta)_{36}\,(\eta\theta)$,	
	3	$(abcdef)_{36_3}\,(dh)$,	8_1
	4–6	$\{(abcdef)_{72}\,(gh)\}$ dim,	6
96	1	$(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta)(\eta\theta)$,	
	2	$(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta\theta)$ cyc,	
	3	$(abcd)$ all $(efgh)_4$,	8_1
	4	$(abcdef)_{48}\,(gh)$,	ʻʻ

Order.	No.	Notation.		
	5	$(\alpha\beta\gamma\delta)$ pos $(\varepsilon\zeta\eta\theta)_8$,		
	6	$\{(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta\theta)_8$ com$\}$ dim,		
	7	$\{(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta\theta)_8$ cyc$\}$ dim,		
	8	$\{(abcd)$ all $(efgh)_8$ pos$\}$ dim,		8_6
	9	$\{(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta\theta)$ all$\}_{4,4}$,		
	10	$\{(\alpha\beta\gamma\delta)$ pos $(\varepsilon\zeta\eta\theta)$ pos$\}_{4,4}(\alpha\varepsilon.\beta\zeta.\gamma\eta.\delta\theta)$,		
	11	$\}(abcd)$ pos $(efgh)$ pos$\}_{4,4}(ae.bg.cf.dh)$,		8_9
	12	$\{(\alpha\beta)(\gamma\delta)(\varepsilon\zeta)(\eta\theta)\}$ pos $(\alpha\gamma\theta.\beta\delta\eta)(\alpha\delta\varepsilon.\beta\gamma\zeta)$,		
120	1	$(\alpha\beta\gamma\delta\varepsilon)_{20}(\zeta\eta\theta)$ all,		
	2	$(abcdef)_{60}(gh)$,	ins.	7_1
	3	$\{(abcdef)_{120}(gh)\}$ pos,	ins.	5
128	1	$(\alpha\beta\gamma\delta)_8(\varepsilon\zeta\eta\theta)_8(\alpha\varepsilon.\beta\zeta.\gamma\eta.\delta\theta)$,		
144	1	$(\alpha\beta\gamma\delta)$ pos $(\varepsilon\zeta\eta\theta)$ pos,		
	2	$(\alpha\beta\gamma\delta\varepsilon\zeta)_{72}(\eta\theta)$,		
168	1	$(abcdefgh)_{168}$,	ins.	7
	2	$A(\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta)_8(\beta\gamma\varepsilon\delta\eta\theta\zeta)$,		
180	1	$(\alpha\beta\gamma\delta\varepsilon)$ pos $(\zeta\eta\theta)$ cyc,	ins.	
192	1	$(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta\theta)_8$,		
	2	$\{(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta\theta)$ all$\}_{4,4}(\alpha\varepsilon.\beta\zeta.\gamma\eta.\delta\theta)$,		
	3	$(\alpha\beta)(\gamma\delta)(\varepsilon\zeta)(\eta\theta)(\alpha\gamma\varepsilon.\beta\delta\zeta)(\gamma\varepsilon\eta.\delta\zeta\theta)$,		
	4	$+\{(\alpha\beta)(\gamma\delta)(\varepsilon\zeta)(\eta\theta)\}$ pos $(\alpha\gamma\varepsilon.\beta\delta\zeta)(\varepsilon\eta.\zeta\theta)$,		
	5	$\pm\{(\alpha\beta)(\gamma\delta)(\varepsilon\zeta)(\eta\theta)\}$ pos $(\alpha\gamma\varepsilon.\beta\delta\zeta)(\alpha\eta\beta\theta)$.		
240	1	$(abdef)_{120}(gh)$,	ins.	7
288	1	$(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta\theta)$ pos,		
	2	$\{(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta\theta)$ all$\}$ pos,		
	3	$(\alpha\beta\gamma\delta)$ pos $(\varepsilon\zeta\eta\theta)$ pos $(\alpha\varepsilon.\beta\zeta.\gamma\eta.\delta\theta)$,		
336	1	$(\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta)_{336}$,	ins.	
360	1	$(\alpha\beta\gamma\delta\varepsilon)$ all $\zeta\eta\theta)$ cyc,	ins.	
	2	$(\alpha\beta\gamma\delta\varepsilon)$ pos $(\zeta\eta\theta)$ all,	ins.	
	3	$\{(\alpha\beta\gamma\delta\varepsilon)$ all $(\zeta\eta\theta)$ all$\}$ pos,	ins.	
384	1	$(\alpha\beta)(\gamma\delta)(\varepsilon\zeta)(\eta\theta)(\alpha\gamma\varepsilon.\beta\delta\zeta)(\gamma\eta.\delta\theta)$,		
576	1	$(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta\theta)$ all,		
	2	$\{(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta\theta)$ all$\}$ pos $(\alpha\varepsilon.\beta\zeta.\gamma\eta.\delta\theta)$,		
	3	$\{(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta\theta)$ all$\}$ pos $(\alpha\varepsilon\beta\zeta.\gamma\eta.\delta\theta)$,		

Order.	No.	Notation.		
720	1	$\{(\alpha\beta\gamma\delta\varepsilon)$ all $(\zeta\eta\theta)$ all,	ins.	
	2	$(\alpha\beta\gamma\delta\varepsilon\zeta)$ pos $(\eta\theta)$	ins.	
	3	$\{(abcdef)$ all $(gh)\}$ pos,	ins.	6
1152	1	$(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta\theta)$ all $(\alpha\varepsilon\,.\,\beta\zeta\,.\,\gamma\eta\,.\,\delta\theta)$,		
1344	1	$(\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta)_{1344}$,	ins.	
1440	1	$(\alpha\beta\gamma\delta\varepsilon\zeta)$ all $(\eta\theta)$,	ins.	
20160	1	$(\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta)$ pos,	ins.	
40320	1	$(\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta)$ all,	ins.	
Total,	200			

EXPLANATIONS

Degree Three.

(abc) all $\equiv 1,\ abc,\ acb,\ ab,\ ac,\ bc.$

(abc) cyc $\equiv 1,\ abc,\ acb.$

*Degree Four.**

$(ac\,.\,bd) \equiv 1,\ ac\,.\,bd.$

$(ab)(cd) \equiv 1,\ ab\,.\,cd,\ ab,\ cd.$

$(abcd)_4 \equiv 1,\ ab\,.\,cd,\ ac\,.\,bd,\ ad\,.\,bc.$

$(abcd)$ cyc $\equiv 1,\ ac\,.\,bd,\ abcd,\ adcb.$

$(abcd)_8 \equiv 1,\ ac\,.\,bd,\ ac,\ bd,\ ab\,.\,cd,\ ad\,.\,bc,\ abcd,\ adcb.$

$(abcd)$ pos $\equiv 1,\ ab\,.\,cd,\ ac\,.\,bd,\ ad\,.\,bc,\ abc,\ acd,\ bdc,\ adb,\ acb,\ bcd,\ abd,\ adc.$

$(abcd)$ all $\equiv (abcd)$ pos $+\ ab,\ cd,\ acbd,\ adbc,\ bc,\ ad,\ acdb,\ abdc,\ ac,\ bd,\ abcd,\ adcb.$

Degree Five.

$(abcde)$ cyc $\equiv 1,\ abcde,\ acebd,\ adbec,\ aedcb.$

(abc) cyc $(de) \equiv 1,\ de,\ abc,\ abc\,.\,de,\ acb,\ acb\,.\,de.$

$\{(abc)$ all $(de)\}$ pos $\equiv 1,\ abc,\ acb,\ ab\,.\,de,\ ac\,.\,de,\ bc\,.\,de$; i. e. the positive substitutions in the direct product of (abc) all and (de).

$(abcde)_{10} \equiv 1,\ abcde,\ acebd,\ adbec,\ aedcb,\ be\,.\,cd,\ ae\,.\,bd,\ ad\,.\,bc,\ ac\,.\,de,\ ab\,.\,ce.$

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*J. A. Serret, Lionville's Journal, 1850, pp. 52, 53.

(abc) all $(de) \equiv 1$, *abc*, *acb*, *ab*, *ac*, *bc*, *de*, *abc.de*, *acb.de*, *ab.de*, *ac.de*, *bc.de*.

$(abcde)_{20}$[*] $\equiv 1$, *abcde*, *acebd*, *adbec*, *aedcb*, *be.cd*, *ae.bd*, *ad.bc*, *ac.de*, *ab.ce*, *bced*, *acbe*, *aecd*, *abdc*, *adeb*, *bdec*, *adce*, *abed*, *aebc*, *acdb*.

Degree Six.

$\{(abcd)_4 (ef)\}$ dim $\equiv 1$, *ab.cd*, *ac.bd.ef*, *ad.bc.ef* or the conjugates obtained by transforming this by (abc) cyc.

$(abc.def)$ all $\equiv 1$, *abc.def*, *acb.dfe*, *ab.de*, *ac.df*, *bc.ef*.

$(abcdef)_6 \equiv 1$, *abc.def*, *acb.dfe*, *ad.bf.ce*, *af.be.cd*, *ae.bd.cf*.

$\{(abcd)_8$ com $(ef)\}$ dim $\equiv 1$, *ac.bd*, *ac*, *bd*, *ab.cd.ef*, *ad.bc.ef*, *abcd.ef*, *adcb.ef*.

$\{(abcd)_8$ cyc $(ef)\}$ dim $\equiv 1$, *ac.bd*, *abcd*, *adcb*, *ac.ef*, *bd.ef*, *ab.cd.ef*, *ad.bc.ef*.

$\{(abcd)_8$ pos $(ef)\}$ dim $\equiv 1$, *ac.bd*, *ab.cd*, *ad.bc*, *ac.ef*, *bd.ef*, *abcd.ef*, *adcb.ef*.

$(abcdef)_{12} \equiv 1$, *abc.def*, *acb.dfe*, *ab.de*, *ac.df*, *bc.ef*, *ad.be.cf*, *aecdbf*, *afbdce*, *ae.bd.cf*, *af.be.cd*, *ad.bf.ce*.

$(abcdef)_{12_2} \equiv 1$, *ac.bd*, *ac.ef*, *bd.ef*, *abe.cdf*, *adf.bec*, *abf.cde*, *ade.bfc*, *aeb.cfd*, *afd.bce*, *afb.ced*, *aed.bcf*.

All the substitutions of $(abcdef)_{24_5}$ and $(abcdef)_{36_3}$ are given by Cole, Bulletin of the New York Mathematical Society, vol. II (1893), p. 185; also in Quarterly Journal of Mathematics, vol. XXVI, p. 372. A complete list of the substitutions in the remaining transitive groups of this degree with the exception of the alternating and the symmetric group is given by Cayley, Quarterly Journal of Mathematics, vol. XXV (1891), pp. 80–85. These groups are also given by Veronese, Annali di Matematica, vol. XI (1883), pp. 176–190.

Degree Seven. Cayley's list (Quarterly Journal, vol. XXV) contains all these groups with the exception of $\{(abcd)$ all (efg) all$\}_{4,1}$ and $(abcdefg)_{168}$. These two groups are given by Cole, Quarterly Journal of Mathematics, vol. XXVI, p. 373. The latter of these two groups is simple and it has received a great deal of attention. Cf. Kirkman, Proceedings of the Literary and Philosophical Society of Manchester, 1863, p. 65; also Moore, Bulletin of the American Mathematical Society, Vol. I (1894), p. 61.

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[*] Lagrange, Oeuvres, vol. III, p. 339.

Degree Eight. The given list by Cayley omits 43 of these groups. Cole's supplementary list (Bulletin of the New York Mathematical Society, vol. II, p. 184) omits two groups, $(abcd\,.\,efgh)$ pos $(af\,.\,be\,.\,cg\,.\,dh)$ and $(abcdefgh)_{1344}$, and it gives a notation for an intransitive group of order 16 which does not exist, Miller, Bulletin of the New York Mathematical Society, vol. III (1894), p. 168. The given group of order 1344 was studied by Jordan, Comptes rendus, vol. LXXIII; Noether, Mathematische Annalen, vol. XV, p. 90; Kirkman, Proceedings of the Literary and Philosophical Society of Manchester, vol. III, p. 150, etc. It is the only compound perfect group of degree 8, and there is no such group of any lower degree, Miller, American Journal of Mathematics, vol. XX, p. 280. Hence, there are six abstract perfect groups which may be represented as substitution groups whose degree does not exceed 8, viz. one group of each of the orders 60, 168, 360, 1344, 2520, and 20,160. $(abcdefgh)_{1344}$ is the holomorph of $A(abcdefgh)_8$. The holomorphs of $B(abcdefgh)_8$, $C(abcdefgh)_8$, $D(abcdefgh)_8$, and $(abcdefgh)$ cyc are $A\{(abcd)_8(efgh)_8\}_{2,2}$ $(ae\,.\,bf\,.\,cg\,.\,dh)$, $M\{(abcd)_8(efgh)_8\}$ dim $(ae\,.\,bf\,.\,cg\,.\,dh)$, $\neq\{(ab)(cd)(ef)(gh)\}$ pos $(ace\,.\,bdf)(agbh)$, and $A\{(abcd)_8(efgh)_8\}_{2,2}(ae\,.\,bf\,.\,cg\,.\,dh)$ respectively. It is interesting to observe that the first and last of these four groups have the same holomorph.

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